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ON CERTAIN PRIORITY QUEUES

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NOTATION

1. *Unit* means service unit or counter at which the server is serving.

2. L.S.T. is abbreviated for Laplace-Steiltjes Transform.

3. $H(x)$ is the distribution function of service times. Its L.S.T. is denoted by $h(s)$ and its first three moments by α, β and γ respectively.

4. $\gamma_1(s)$ denotes the L.S.T. of the distribution of busy period of an $M|G|1$ queue with service time distribution $H_1(x)$.

5. $\gamma(s)$ denotes the L.S.T. of the distribution of busy period of the whole system (All the service units considered together).

6. The convolution of two distribution functions $F(x)$ and $G(x)$, $0 \leq x \leq \infty$, is denoted by:

$$F*G(x) = \int_0^x F(x-u)\, dG(u)$$

The m-fold convolution of F is denoted by:

$$F^{(m)}(x) = F*F^{(m-1)}(x)$$

7. $U(x)$ is the unit distribution:

$$U(x) = 0 \text{ if } x < 0$$
$$= 1 \text{ if } x \geq 0$$

8. δ_{ij} is the Kronecker delta defined by:

$$\delta_{ij} = 0 \text{ if } i \neq j$$
$$= 1 \text{ if } i = j$$

9. For referring the equations we use the following convention: (n) means the n-th equation of the present chapter and (m·n) means the n-th equation of the m-th chapter.

CHAPTER I

A SINGLE SERVER TANDEM QUEUE WITH NON-ZERO SWITCHING IN UNIT 1

1. Concepts and Definitions

In this chapter we consider a queueing process with two service units, unit 1 and unit 2, and a single server. The server attends to the two units alternately according to some switching rule. A switching rule [Neuts and Yadin, 1968] is a rule describing how the server changes from one unit to the other. The server may change from one unit to the other either by a non-zero switching rule or by a zero switching rule. By a non-zero switching rule the server continues to serve in a unit until some upper number of consecutive services have been completed and then he switches to the other unit. By a zero switching rule the server stays in a unit until the queue in it becomes empty and then he switches to the other unit.

In this chapter we discuss a non-zero switching rule for unit 1 and zero switching rule for unit 2. The zero switching rule for unit 1 is dealt in the next chapter.

We say that two units are in tandem when the output of the first unit is the input to the second. It is assumed that

customers arrive in unit 1 in accordance with a Poisson process of density λ. The input for unit 2 is those who have completed service in unit 1.

The durations of the successive service times in units 1 and 2 are identically distributed independent positive random variables with distribution functions $H_1(\cdot)$ and $H_2(\cdot)$ respectively. Further the service times are independent of the arrival times.

In the case of non-zero switching the server starts in unit 1 at time t = 0 and continues to serve in it until he has given k services without interruption or until the queue becomes empty, whichever comes first. k is a positive integer which we will call as the switching parameter. The time interval spent without interruption in unit 1 is called a 1-task. Similarly we define a 2-task. A 1-task followed by a 2-task both together will be called as a cycle of tasks.

The customers who have completed service in unit 1 queue up in front of unit 2. The server after completing the 1-task switches to unit 2 and serves there until the queue in it becomes empty. After finishing the task in unit 2 the server switches back to unit 1 and continues the process.

When k=1, we obtain simply an M/G/1 queue with service time distribution $H_1 * H_2(\cdot)$

2. Distribution of Busy Period

The server begins in unit 1 at $t=0$ and serves between the two units alternately according to some switching rule. The time required for both the units to become empty simultaneously for the first time is called a busy period of the system.

Suppose that there is a Poisson input of density λ in unit 1 and that the service time distributions of the two units are $H_1(\cdot)$ and $H_2(\cdot)$. Then since the distribution of busy period does not depend upon the order in which the customers are served [Welch (1965)], the distribution of busy period of the model defined above is equivalent to the distribution of busy period of an M/G/1 queue with input rate λ and service time distribution the convolution $H_1 * H_2(\cdot)$. Hence from the classical results of an M/G/1 queue [Takács 1962, p. 47] that if $\gamma(s)$ is the Laplace Stieltjes Transform (L.S.T.) of the distribution of busy period then $\gamma(s)$ is the unique root in the unit disk $|z| < 1$ of the equation

(1) $$z = h_1(s + \lambda - \lambda z)\, h_2(s + \lambda - \lambda z),$$

The expected length of busy period is given by:

(2) $$-\gamma'(0+) = \frac{\alpha_1 + \alpha_2}{1-\lambda\alpha_1-\lambda\alpha_2} \quad \text{if } 1-\lambda\,\alpha_1 - \lambda\,\alpha_2 > 0,$$

$$= \infty \qquad \text{if } 1-\lambda\,\alpha_1 - \lambda\,\alpha_2 = 0,$$

3. The Basic Imbedded Semi-Markov Process and its Transition Probabilities

Macroscopically the queueing process consists of busy periods alternating with idle periods. Each busy period consists of a random number of alternating 1-tasks and 2-tasks. Every busy period can be decomposed into a random number of cycles of tasks.

Here we assume that at t=0 there are $i > 0$ customers in unit 1 and none in unit 2. In the case $i=0$ the process starts with an idle period.

Let us define the sequence of random variables T_o, T_1, $T_2, \ldots\ldots$, where $T_o = 0$ and T_n is the duration of the nth cycle of tasks, $n=1,2,\ldots.$ Let ξ_n denote the number of customers in the system at the end of the nth cycle, $n=1,2,\ldots.$ and $\xi_o = i$.

It follows from the regenerative properties of the input and service processes that the bivariate sequence of random variables

$$(3) \qquad \left\{\xi_n, T_n, n \geq 0\right\}$$

is a Semi-Markov sequence with state space $\{0,1,2,\ldots\ldots\}$.

We recall the definition of Semi-Markov sequence. Consider a double sequence of random variables $\left\{(J_n,X_n), n=0,1,2,\ldots\right\}$ defined on a complete probability space and such that:

(i) (4) $\quad P\{X_0 = 0\} = 1,$

(ii) $\quad P\{J_0=k\}=a_k$, where $a_k \geq 0$, $\sum_{k\in I} a_k = 1$, and I is the state space

(iii) $\quad P\{J_n=k,\ X_n \leq x \mid J_0, J_1, \ldots, J_{n-1}, X_1, X_2, \ldots, X_{n-1}\}$

$$= P\{J_n=k,\ X_n \leq x \mid J_{n-1}\} = Q_{J_{n-1}k}(x),$$

for $n=1,2,\ldots\ldots$

then the process $\{(J_n, X_n),\ n \geq 0\}$ is called a Semi-Markov sequence. The functions $Q_{ij}(x)$, $i,j=1,2,\ldots.$ are mass functions which are non-decreasing and they satisfy:

$$Q_{ij}(x) = 0 \quad \text{for } x \leq 0,$$

$$Q_{ij}(\infty) = P_{ij}, \quad i,j=1,2,\ldots$$

where (P_{ij}) is the transition matrix of the Markov chain $\{J_n,\ n \geq 0\}$. For further details of Semi-Markov sequences we refer to Pyke (1961), Neuts (1966).

To study the transition probabilities of the Semi-Markov sequence we first define an auxiliary probability function $G_{ij}^{(n)}(x)$.

Let us define:

(5a) $\quad G_{ij}^{(0)}(x) = \delta_{ij}\ U(x),$

where δ is the Kronecker delta and $U(\cdot)$ is the distribution degenerate at zero. For $n \geq 1$, $G_{ij}^{(n)}(x)$ is the probability that, in an $M|G|1$ queue of input rate λ and service time distribution $H_1(\cdot)$ the initial busy period involves at least n services, that the n-th service is completed before

time x and that at the end of the n-th service there are j customers waiting, given that there were i customers initially. Then for $i \geq 1$:

(5b) $$G_{ij}^{(1)}(x) = \int_0^x e^{-\lambda y} \frac{(\lambda y)^{j-i+1}}{(j-i+1)!} \, dH_1(y) \, ,$$

(5c) $$G_{ij}^{(n+1)}(x) = \sum_{\nu=1}^{j+1} \int_0^x G_{i\nu}^{(n)}(x-y) e^{-\lambda y} \frac{(\lambda y)^{j-\nu+1}}{(j-\nu+1)!} \, dH_1(y) \, , \quad n \geq 1,$$

Let $g_{ij}^{(n)}(s)$ be the L.S.T. of $G_{ij}^{(n)}(x)$ and:

(6) $$g_i^{(n)}(s,z) = \sum_{j=0}^{\infty} g_{ij}^{(n)}(s) z^j \quad , \quad |z| \leq 1, \; n \geq 0 \, ,$$

Then:

(7) $$\begin{aligned} g_i^{(0)}(s,z) &= z^i \, , \\ g_i^{(1)}(s,z) &= z^{i-1} \, h_1(s+\lambda-\lambda z) \, , \\ g_i^{(n+1)}(s,z) &= \frac{h_1(s+\lambda-\lambda z)}{z} \left[g_i^{(n)}(s,z) - g_i^{(n)}(s,0) \right], \quad n \geq 1, \end{aligned}$$

where $h_1(\cdot)$ is the L.S.T. of $H_1(\cdot)$.

Successive substitution yields:

(8a)

$$g_i^{(n+1)}(s,z) = \left[\frac{h_1(s+\lambda-\lambda z)}{z}\right]^{n+1} z^i - \sum_{\nu=1}^{n}\left[\frac{h_1(s+\lambda-\lambda z)}{z}\right]^{\nu} g_i^{(n-\nu+1)}(s,0)$$

(8b)
$$= \left[\frac{h_1(s+\lambda-\lambda z)}{z}\right]^{n+1} z^i - \sum_{\nu=1}^{n}\left[\frac{h_1(s+\lambda-\lambda z)}{z}\right]^{n-\nu+1} g_i^{(\nu)}(s,0),$$

≥ 1 ,

From the definition of $G_{ij}^{(n)}(x)$ it follows that:

(9) $$g_i^{(n)}(s,0) = 0 \qquad \text{for } i > n ,$$

Hence from (8):

(10) $$g_i^{(n)}(s,z) = z^{i-n}\, h_1^n(s+\lambda-\lambda z) \quad \text{for } i \geq n \geq 0 ,$$

A Summary of Known Results

The properties of the probability functions $G_{ij}^{(n)}(x)$, already known, may be summarized as follows: For proofs of these we refer to Takács (1960), Neuts (1968b)

Lemma 1.1

If $G_1(x)$ is the distribution of busy periods for an $M|G|1$ queue with input rate λ and service time distribution $H_1(\cdot)$ and $\gamma_1(s)$ its L.S.T., then:

(11) $$\sum_{n=1}^{\infty} g_{io}^{(n)}(s) = \gamma_1^i(s) , \; i \geq 1,$$

<u>Lemma 1.2</u>

If $\gamma_1(s,\omega) = \sum_{n=1}^{\infty} g_1^{(n)}(s,0)\ \omega^n \quad , \ |\omega| \leq 1$,

then:

$$\sum_{n=1}^{\infty} g_i^{(n)}(s,0)\ \omega^n = \gamma_1^i(s,\omega)\ , \quad i \geq 1\ , \tag{12}$$

For $\omega = 1$, $\gamma_1(s,1) = \gamma_1(s)$ and then lemma 1.2 reduces to lemma 1.1.

<u>Lemma 1.3</u>

If $|z| \leq 1$, $|\omega| \leq 1$ and $i \geq 1$ then:

$$\sum_{n=0}^{\infty} \sum_{j=1}^{\infty} g_{ij}^{(n)}(s) z^j\ \omega^n = \frac{z[z^i - \gamma_1^i(s,\omega)]}{z - \omega h_1(s+\lambda-\lambda z)} \quad , \tag{13}$$

For $\omega = 1$ one may rewrite (13) as:

$$\sum_{n=0}^{\infty} \sum_{j=1}^{\infty} g_{ij}^{(n)}(s) z^j = \frac{z[z^i - \gamma_1^i(s)]}{z - h_1(s+\lambda-\lambda z)}\ , \tag{14}$$

<u>Lemma 1.4</u>

If $R(s) \geq 0$ and $|\omega| \leq 1$ then $z = \gamma_1(s,\omega)$ is a root of Takács' functional equation:

$$z = \omega\, h_1(s+\lambda-\lambda z)\ , \qquad\qquad |z| \leq 1\ , \tag{15}$$

Further $z = \gamma_1(s,\omega)$ is the <u>only</u> root of this equation in the unit circle $|z| < 1$ if $R(s) \geq 0$ and $|\omega| < 1$ or $R(s) > 0$ and $|\omega| \leq 1$ or $R(s) \geq 0$, $|\omega| \leq 1$ and $1 - \lambda\, \alpha_1 < 0$.

<u>Lemma 1.5</u>

In lemma 1.4 taking $\omega = 1$ we get that for $R(s) \geq 0$, $z = \gamma_1(s)$ is a root of the equation:

$$z = h_1(s+\lambda-\lambda z) \ , \tag{16}$$

Further $\theta = \gamma_1(0)$ is the smallest positive real root of the equation:

$$\theta = h_1(\lambda - \lambda\theta) \ , \tag{17}$$

and if $1-\lambda\, \alpha_1 < 0$ then $\theta < 1$ and if $1 - \lambda\, \alpha_1 \geq 0$ then $\theta=1$.

From lemmas 1.4 and 1.5 it follows that:

$$\begin{aligned} \gamma_1'(0+) &= \frac{-\alpha_1}{1-\lambda\, \alpha_1} && \text{if } 1 - \lambda\, \alpha_1 > 0 \ , \\ &= \infty && \text{if } 1 - \lambda\, \alpha_1 = 0, \end{aligned} \tag{18}$$

$$\gamma_1''(0+) = \frac{\beta_1}{(1-\lambda\, \alpha_1)^3} \quad \text{if } 1 - \lambda\, \alpha_1 > 0 \ , \tag{19}$$

If $\gamma_1(0,\omega) = f(\omega)$ then

$$\begin{aligned} f'(1) &= \frac{1}{1-\lambda\, \alpha_1} && \text{if } 1 - \lambda\, \alpha_1 > 0 \ , \\ &= \infty && \text{if } 1 - \lambda\, \alpha_1 = 0, \end{aligned} \tag{20}$$

(21) $$f''(1) = \frac{2\lambda\alpha_1}{(1-\lambda\alpha_1)^2} + \frac{\lambda^2\beta_1}{(1-\lambda\alpha_1)^3} \text{ if } 1-\lambda\alpha_1 > 0,$$

Now we define the transition probabilities of the Semi-Markov sequence $\{\xi_n, T_n, n \geq 0\}$ defined in (3) as:

(22) $$Q_{ij}(x) = P\{\xi_n = j, T_n \leq x \mid \xi_{n-1} = i\}$$

For $i > 0$ and $j \geq 0$,

(23)

$$Q_{ij}(x) = \sum_{\nu=0}^{j} \underset{(u)(v)}{\int_0^x \int_u^x} dG_{i\nu}^{(k)}(u) e^{-\lambda(v-u)} \frac{[\lambda(v-u)]^{j-\nu}}{(j-\nu)!} d_v H_2^{(k)}(v-u)$$

$$+ \sum_{r=i}^{k-1} \underset{(u)(v)}{\int_0^x \int_u^x} dG_{io}^{(r)}(u) e^{-\lambda(v-u)} \frac{[\lambda(v-u)]^{j}}{j!} d_v H_2^{(r)}(v-u),$$

where $H^{(n)}(\cdot)$ is the n-fold convolution of $H(\cdot)$. The second term on the right hand side of the above expression vanishes for $i \geq k$.

If $q_{ij}(s)$ is the L.S.T. of $Q_{ij}(x)$ and

(24) $$q_i(s,z) = \sum_{j=0}^{\infty} q_{ij}(s)\, z^j, \quad |z| \leq 1,$$

then:

(25)

$$q_{ij}(s) = \sum_{\nu=0}^{j} g_{i\nu}^{(k)}(s) \int_0^\infty e^{-(s+\lambda)x} \frac{(\lambda x)^{j-\nu}}{(j-\nu)!} \, d\, H_2^{(k)}(x)$$

$$+ \sum_{r=i}^{k-1} g_{io}^{(r)}(s) \int_0^\infty e^{-(s+\lambda)x} \frac{(\lambda x)^{j}}{j!} \, d\, H_2^{(r)}(x) ,$$

for $i > 0$ and $j \geq 0$,

(26a)

$$q_i(s,z) = g_i^{(k)}(s,z)h_2^k(s+\lambda-\lambda z) + \sum_{r=i}^{k-1} g_i^{(r)}(s,0)h_2^r(s+\lambda-\lambda z),$$

for $i < k$,

(26b) $= g_i^{(k)}(s,z)\, h_2^k(s+\lambda-\lambda z)$, for $i \geq k$,

From (9) and (26) it follows that:

(27) $$q_i(s,0) = \sum_{r=i}^{k} g_i^{(r)}(s,0)h_2^r(s+\lambda) \text{ if } i \leq k,$$

$$= 0 \qquad\qquad \text{if } i > k \;;$$

Next we introduce the taboo probabilities ${}_oQ_{ij}^{(n)}(x)$ defined by:

(28)

$${}_oQ_{ij}^{(n)}(x) = P\left\{T_o+T_1+\ldots+T_n \leq x, \xi_n=j, \xi_\nu \neq 0 \text{ for } \nu=1,\ldots,n-1 \;\middle|\; \xi_o = i\right\}, \quad n \geq 1 ,$$

$${}_oQ_{ij}^{(o)}(x) = \delta_{ij}\, \cup(x)$$

That is, ${}_oQ_{ij}^{(n)}(x)$ is the probability that a busy period has at least n cycles of tasks, that the n-th cycle ends not later than x and when it does j customers are waiting, given that i customers were in unit 1 at t=0.

From the definition it follows that:

(29) $${}_oQ_{ij}^{(1)}(x) = Q_{ij}(x) \quad ,$$

and

$$ {}_{o}Q_{ij}^{(n+1)}(x) = \sum_{\nu=1}^{j+k} \int_{o}^{x} {}_{o}Q_{i\nu}^{(n)}(x-u)\, d\, Q_{\nu j}(u), \quad n \geq 1 , $$

Since each cycle of tasks can have atmost k services in units 1 and 2, in the above formula ν can be atmost $j+k$.

Let ${}_{o}q_{ij}^{(n)}(s)$ be the L.S.T. of ${}_{o}Q_{ij}^{(n)}(\cdot)$ and:

(30a) $$ {}_{o}r_{ij}(s) = \sum_{n=0}^{\infty} {}_{o}q_{ij}^{(n)}(s), $$

(30b) $$ {}_{o}m_{ij}(s) = \sum_{n=1}^{\infty} {}_{o}q_{ij}^{(n)}(s) , $$

(31a) $$ {}_{o}r_{i}(s,z) = \sum_{j=0}^{\infty} {}_{o}r_{ij}(s)\, z^{j}, $$

(31b) $$ {}_{o}m_{i}(s,z) = \sum_{j=0}^{\infty} {}_{o}m_{ij}(s)\, z^{j}, \qquad |z| \leq 1,\ R(s) > 0, $$
$$ \text{or } |z| < 1,\ R(s) \geq 0, $$

(32) $$ {}_{o}q_{i}^{(n)}(s,z) = \sum_{j=0}^{\infty} {}_{o}q_{ij}^{(n)}(s)\, z^{j} , $$

Then:

$$(33) \qquad {}_oq_{ij}^{(o)}(s) = \delta_{ij} \quad ,$$

$${}_oq_{ij}^{(n)}(s) = \sum_{\nu=1}^{j+k} {}_oq_{i\nu}^{(n-1)}(s)\, q_{\nu j}(s) \;, \quad n \geq 1,$$

$$(34) \qquad {}_oq_i^{(o)}(s,z) = z^i \;,$$

$${}_oq_i^{(n)}(s,z) = \sum_{\nu=1}^{\infty} {}_oq_{i\nu}^{(n-1)}(s)\, q_\nu(s,z) \;, \quad n \geq 1,$$

$${}_om_i(s,z) = q_i(s,z) + \sum_{\nu=1}^{\infty} {}_om_{i\nu}(s)\, q_\nu(s,z) \;,$$

Note that $\sum_{n=1}^{\infty} {}_oq_{io}^{(n)}(s) = {}_om_i(s,0)$ is the L.S.T. of the distribution of busy period of an $M|G|1$ queue with a Poisson input of rate λ and service time distribution $H_1 * H_2(\cdot)$. That is:

$$(35) \qquad {}_om_i(s,0) = \gamma^i(s) \quad , \; i \geq 1 \;.$$

More properties of the taboo probabilities ${}_oQ_{ij}^{(n)}(\cdot)$ are studied in the zero switching case.

4. The Joint Distribution of Queue length and Virtual Waiting time

Virtual Waiting time:

The virtual waiting time at time t is defined as the length of time a (virtual) customer arriving at t has to wait before beginning service in unit 1. For the non-zero switching case the virtual waiting time at time t will be denoted by

$\eta_1^{(k)}(t)$, where k is the switching parameter defined earlier.

Queue length:

The number of individuals in the system at time t still requiring some service in unit 1 is defined as the queue length at time t. For the non-zero switching case this quantity will be denoted by $\xi^{(k)}(t)$.

Let $\theta_{ij}(t,x)$ be the joint distribution of the queue length $\xi^{(k)}(t)$ and virtual waiting time $\eta_1^{(k)}(t)$, given that at t=0 there are $i \geq 1$ customers. That is:

(36) $$\theta_{ij}(t,x) = P\left\{\xi^{(k)}(t) = j,\ \eta_1^{(k)}(t) \leq x \mid \xi^{(k)}(0) = i\right\},$$

Further let for $i \geq 1$,

(37)

$$\phi_{ij}(t,x) = P\left\{\xi^{(k)}(t)=j,\ 0 < \eta_1^{(k)}(t) \leq x,\ \eta_1^{(k)}(\tau)\neq 0 \text{ for all } \tau \in (0,t] \mid \xi^{(k)}(0) = i\right\},$$

Formula (36) can be written in terms of (37) as:

(38)

$$\theta_{ij}(t,x) = \phi_{ij}(t,x) + \int_0^t \phi_{1j}(t-u,x)\, d\, M_1(u)$$

$$+ P\left\{\xi^{(k)}(t)=j,\ \eta_1^{(k)}(t)=0 \mid \xi^{(k)}(0) = i\right\} U(x),$$

where $M_1(\cdot)$ is the renewal function of the general renewal process formed by the beginnings of busy periods,

$$U(x) = \begin{cases} 0 \text{ for } x < 0 \\ 1 \text{ for } x \geq 0 \end{cases}$$

To obtain the equation (38), consider the event on the right hand side of (36) which can be split into three mutually exclusive events:

(i) The time t falls in the initial busy period, $\xi^{(k)}(t) = j$ and $0 < \eta_1^{(k)}(t) \leq x$, given that $\xi^{(k)}(0) = i$.

(ii) The time t does not fall in the initial busy period but in some other busy period which started at time u $(0 < u \leq t)$ with a single customer, $\xi^{(k)}(t) = j$ and $0 < \eta_1^{(k)}(t) \leq x$, given that $\xi^{(k)}(0) = i$.

(iii) The server is idle at time t (that is $\eta_1^{(k)}(t) = 0$), given that $\xi^{(k)}(0) = i$.

The probabilities of these three events give respectively the three terms on the right hand side of (38).

For $i \geq 1$, let $\psi_{ij}(t,x)$ be the probability that at t the original cycle of task has not yet ended and that $\xi^{(k)}(t) = j$, $0 < \eta_1^{(k)}(t) \leq x$ and $\eta_1^{(k)}(\tau) \neq 0$ for all $\tau \in (0,t]$, given that at t=0 the service started in unit 1 with i customers. Then:

$$(39) \qquad \psi_{ij}(t,x) = \sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} \int_{0}^{t} d_{(u)} \, {}_{0}Q_{i\nu}^{(n)}(u) \, \psi_{\nu j}(t-u,x) \, ,$$

This formula is obtained from the fact that at time t the server is serving in the (n+1)th cycle (n=0,1,....) of the initial busy period. The n-th cycle ended between u and u+du ($0 \le u < t$) leaving $\nu(=1,2,\ldots)$ customers in the system.

We define the following transforms for $R(s) > 0$, $R(\zeta) \ge 0$ and $|z| \le 1$:

$$\theta^*_{ij}(t,s) = \int_0^\infty e^{-sx}\, d\,\theta_{ij}(t,x) \,,$$

$$\theta^{**}_{ij}(\xi,s) = \int_0^\infty e^{-\xi t}\, \theta^*_{ij}(t,s)\, dt,$$

$$\theta_i(\xi,s,z) = \sum_{j=0}^{\infty} \theta^{**}_{ij}(\xi,s)\, z^j \,,$$

$$\Phi^*_{ij}(t,s) = \int_0^\infty e^{-sx}\, d\,\Phi_{ij}(t,x) \,,$$

$$\Phi^{**}_{ij}(\xi,s) = \int_0^\infty e^{-\xi t}\, \Phi^*_{ij}(t,s)\, dt \,,$$

$$\Phi_i(\zeta,s,z) = \sum_{j=0}^{\infty} \Phi^{**}_{ij}(\zeta,s) z^j \,,$$

$$\psi^*_{ij}(t,s) = \int_0^\infty e^{-sx}\, d\,\psi_{ij}(t,x) \,,$$

$$\psi^{**}_{ij}(\xi,s) = \int_0^\infty e^{-\xi t}\, \psi^*_{ij}(t,s)\, dt,$$

$$\psi_i(\zeta,s,z) = \sum_{j=0}^{\infty} \psi^{**}_{ij}(\zeta,s)\, z^j \,,$$

$$m_1(\zeta) = \int_0^\infty e^{-\xi t}\, d\,M_1(t) \,,$$

<u>Lemma 1.6</u>

For $R(s) > 0$, $R(\xi) \geq 0$ and $|z| \leq 1$, the transform $\psi_i(\xi,s,z)$ is given by:

(40)

$$\psi_i(\xi,s,z) = \frac{1}{k}\sum_{m=0}^{k-1}\Big[h_2^{k-1}(s)\,[\omega_m h_2(s)-1][\xi+\lambda-s-\lambda\omega_m z h_1(s)h_2(s)]$$

$$\cdot\,[z h_1(s)-h_1(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))]\Big]^{-1}$$

$$\cdot\,\Big[\omega_m[h_2^k(s)-1]\Big\{z[h_1(s)-h_1(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))][\omega_m z h_1(s)h_2(s)]^i$$

$$-\,[z h_1(s)-h_1(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))]\,q_i(\xi,\omega_m z h_1(s)h_2(s))$$

$$+\,(z-1)h_1(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))h_2^k(s)g_i^{(k)}(\xi,\omega_m z h_1(s)h_2(s))$$

$$+\sum_{\nu=1}^{k-1}[z(1-\omega_m^\nu)h_1(s)+(\omega_m^\nu z-1)h_1(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))]h_2^\nu(s)g_i^{(\nu)}(\xi,0)\Big\}\Big],$$

where $\omega_0, \omega_1, \ldots, \omega_{k-1}$ are the k-th roots of unity.

<u>Proof:</u>

The probability $\psi_{ij}(t,x)$ is given in terms of the probabilities $G_{i\nu}^{(n)}(u)$ by:

(41)

$$\psi_{ij}(t,x) = \sum_{\nu=0}^{j} \int_{o}^{t} \int_{t}^{t+x} \int_{v}^{t+x} dG_{i\nu}^{(k)}(u)\ e^{-\lambda(t-u)}\ \frac{[\lambda(t-u)]^{j-\nu}}{(j-\nu)!}$$
$$(u)(v)\ (v_1)$$

$$\cdot\ d_v H_2^{(k)}(v-u) d_{v_1} (H_1^{(j)} * H_2^{([\frac{j}{k}]k)})\ (v_1 - v)$$

$$+ \sum_{r=1}^{k-1} \int_{c}^{t} \int_{t}^{t+x} \int_{v}^{t+x} dG_{io}^{(r)}(u)\ e^{-\lambda(t-u)}\ \frac{[\lambda(t-u)]^{j}}{j!}$$
$$(u)(v)\ (v_1)$$

$$\cdot\ d_v H_2^{(r)}(v-u) d_{v_1} (H_1^{(j)} * H_2^{([\frac{j}{k}]k)})(v_1 - v)$$

$$+ \sum_{r=0}^{k-1} \sum_{\nu=1}^{j} \int_{o}^{t} \int_{t}^{t+x} \int_{v}^{t+x} dG_{i\nu}^{(r)}(u)\ e^{-\lambda(t-u)}\ \frac{[\lambda(t-u)]^{j-\nu}}{(j-\nu)!}$$
$$(u)(v)\ (v_1)$$

$$\cdot\ d_v H_1(v-u) d_{v_1} (H_1^{(j-1)} * H_2^{([\frac{j+r}{k}]k)})\ (v_1 - v)\ ,$$

where $[\frac{a}{k}]$ is the greatest integer not exceeding $\frac{a}{k}$.

<u>The first term</u> is obtained by assuming that the server is performing a 2-task at t and that the previous 1-task consisted of k services. The cycle of tasks in which the server is serving at t started with i customers in unit 1 at t=0. The 1-task ends after k services leaving ν customers in unit 1 at time u. The number of arrivals between times u and t is $j-\nu$ so that at t there are j

customers in unit 1. The services of the j customers start after the completion of the 2-task in progress at t. Let this 2-task end at time v. If j is a multiple of k, say mk, where m is a positive integer, then the (virtual) customer eneters service after the service completion of j customers in both the units. That is, after the completion of m cycles of tasks, each cycle consisting of k services in each unit. If j is not a multiple of k, say mk+r $(0 < r < k)$, then the (virtual) customer enters service after the completion of m cycles of tasks together with a further service completion of r customers in unit 1. Let this service completion occur at time v_1. Now we integrate and sum over all choices of ν, u, v, v_1.

The second term is obtained by assuming again that the server is performing a 2-task at t and that the previous 1-task consisted of only r $(< k)$ services, leaving the unit 1 empty at the end of the 1-task at time u.

The last term is obtained by assuming that the server is performing a 1-task at t. The cycle of tasks current at t started with i customers in unit 1 at t=0 and r $(\leq k -1)$ service completions are made before t. The last service completion before t occurs at time u at which there are ν customers waiting in unit 1. The number of arrivals between times u and t is $j-\nu$. Now there are j-1 customers, excepting the customer in service, at t in unit 1. Let the customer

who is in service at t complete his service at time v and let the service completion of the j-l customers occur in unit 1 at time v_1. Finally we sum over all chcices of r, ν, u, v, v_1.

Taking the transform of (41) results in:

(42)

$$\psi^{**}_{ij}(\xi,s) = \sum_{\nu=0}^{j} g_{i\nu}^{(k)}(\zeta)h_1^j(s)h_2^{[\frac{j}{k}]k}(s)\int_0^\infty e^{-sv}\int_0^v e^{-(\xi+\lambda-s)t}\,\frac{(\lambda t)^{j-\nu}}{(j-\nu)!}\,dt$$

$$\cdot\; d\, H_2^{(k)}(v)$$

$$+\sum_{r=1}^{k-1} g_{io}^{(r)}(\xi)h_1^j(s)h_2^{[\frac{j}{k}]k}(s)\int_0^\infty e^{-sv}\int_0^v e^{-(\zeta+\lambda-s)t}\,\frac{(\lambda t)^j}{j!}\,dt$$

$$\cdot\; d\, H_2^{(r)}(v)$$

$$+\sum_{r=0}^{k-1}\sum_{\nu=1}^{j} g_{i\nu}^{(r)}(\zeta)h_1^{j-1}(s)h_2^{[\frac{j+r}{k}]k}(s)\int_0^\infty e^{-sv}\int_0^v e^{-(\zeta+\lambda-s)t}\,\frac{(\lambda t)^{j-\nu}}{(j-\nu)!}\,dt$$

$$\cdot\; d\, H_1(v)\;,$$

Hence:

(43)

$$\psi_i(\xi,s,z) = \sum_{\nu=0}^{\infty} g_{i\nu}^{(k)}(\xi)[zh_1(s)]^{\nu}\int_0^{\infty} e^{-sv}\int_0^{v} e^{-(\xi+\lambda-s)t}$$

$$\cdot \sum_{j=\nu}^{\infty} \frac{[\lambda zh_1(s)t]^{j-\nu}}{(j-\nu)!} \; h_2^{[\frac{j}{k}]k}(s)dt\,dH_2^{(k)}(v)$$

$$+ \sum_{r=i}^{k-1} g_{io}^{(r)}(\xi)\int_0^{\infty} e^{-sv}\int_0^{v} e^{-(\xi+\lambda-s)t} \sum_{j=0}^{\infty} \frac{[\lambda zh_1(s)t]^{j}}{j!}$$

$$\cdot\; h_2^{[\frac{j}{k}]k}(s)\; dt\; d\; H_2^{(r)}(v)$$

$$+ \sum_{r=0}^{k-1} \sum_{\nu=1}^{\infty} g_{i\nu}^{(r)}(\xi)z^{\nu}h_1^{\nu-1}(s)\int_0^{\infty} e^{-sv}\int_0^{v} e^{-(\xi+\lambda-s)t}$$

$$\cdot \sum_{j=\nu}^{\infty} \frac{[\lambda zh_1(s)t]^{j-\nu}}{(j-\nu)!}\; h_2^{[\frac{j+r}{k}]k}(s)dt\; dH_1(v),$$

Taking the summations inside the integrals is justified by Lebesgue Dominated Convergence Theorem. In (43) to sum the series inside the integrals we use the theorem in Appendix A, by taking $y = h_2(s) \neq 1$ for $R(s) > 0$. Then:

$$\psi_i(\xi,s,z) = \frac{1}{k}\sum_{m=0}^{k-1}\Big[h_2^{k-1}(s)[\omega_m h_2(s)-1][\xi+\lambda-s-\lambda\omega_m z h_1(s)h_2(s)]\Big]^{-1}$$

$$\cdot\Big[\omega_m[h_2^k(s)-1]\Big\{\sum_{\nu=0}^{\infty} g_{i\nu}^{(k)}(\xi)[\omega_m z h_1(s)h_2(s)]^{\nu}$$

$$\cdot[h_2^k(s)-h_2^k(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))]$$

$$+\sum_{r=1}^{k-1} g_{io}^{(r)}(\xi)[h_2^r(s)-h_2^r(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))]$$

$$+\sum_{r=0}^{k-1}\sum_{\nu=1}^{\infty} g_{i\nu}^{(r)}(\xi)[\omega_m z h_1(s)h_2(s)]^{\nu}[\omega_m h_2(s)]^{r}$$

$$\cdot[1-h_1(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))|h_1(s)]\Big\}\Big]$$

(44)

$$= \frac{1}{k}\sum_{m=0}^{k-1}\Big[h_2^{k-1}(s)[\omega_m h_2(s)-1][\xi+\lambda-s-\lambda\omega_m z h_1(s)h_2(s)]\Big]^{-1}$$

$$\Big[\omega_m[h_2^k(s)-1]\Big\{g_i^{(k)}(\xi,\omega_m z h_1(s)h_2(s))$$

$$\cdot[h_2^k(s)-h_2^k(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))]$$

$$+\sum_{r=1}^{k-1} g_i^{(r)}(\xi,0)[h_2^r(s)-h_2^r(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))]$$

$$+\sum_{r=0}^{k-1}[\omega_m h_2(s)]^r[g_i^{(r)}(\xi,\omega_m z h_1(s)h_2(s))-g_i^{(r)}(\xi,0)]$$

$$\cdot[1-h_1(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))|h_1(s)]\Big\}\Big],$$

Formula (26) gives:

$$g_i^{(k)}(\xi,\omega_m z h_1(s)h_2(s))h_2^k(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))$$

$$+\sum_{r=i}^{k-1} g_i^{(r)}(\xi,0)h_2^r(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))$$

$$(45)\qquad = q_i(\xi,\omega_m z h_1(s)h_2(s)) \ ,$$

Again, using (8b) we have:

$$g_i^{(r)}(\xi,\omega_m z h_1(s)h_2(s))-g_i^{(r)}(\xi,0) = \left[\frac{h_1(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))}{\omega_m z h_1(s)h_2(s)}\right]^r$$

$$\cdot\left[\omega_m\ z\ h_1(s)\ h_2(s)\right]^i$$

$$-\sum_{\nu=1}^{r}\left[\frac{h_1(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))}{\omega_m z h_1(s)h_2(s)}\right]^{r-\nu} g_i^{(\nu)}(\xi,0) \ ,$$

Using this, the last term in (44) becomes after simplification:

(46)

$$\sum_{r=0}^{k-1}\left[\omega_m h_2(s)\right]^r\left[g_i^{(r)}(\xi,\omega_m z h_1(s)h_2(s))-g_i^{(r)}(\xi,0)\right]$$

$$\cdot\left[1-h_1(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))\ |\ h_1(s)\right]$$

$$=\frac{z\left[h_1(s)-h_1(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))\right]}{z h_1(s)-h_1(\xi+\lambda-\lambda\omega_m z h_1(s)h_2(s))}\Big\{[\omega_m z h_1(s)h_2(s)]^i$$

$$-\sum_{\nu=1}^{k-1}[\omega_m h_2(s)]^\nu g_i^{(\nu)}(\xi,0)-g_i^{(k)}(\xi,\omega_m z h_1(s)h_2(s))h_2^k(s)\Big\} \ ,$$

Substituting (45) and (46) in (44) and simplifying we prove the lemma.

Lemma 1.7

For $R(s) > 0$, $R(\xi) \geq 0$ and $|z| \leq 1$ the transform $\Phi_i(\xi,s,z)$ is given by

$$(47) \qquad \Phi_i(\xi,s,z) = \sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} {}_oq_{i\nu}^{(n)}(\xi)\psi_\nu(\xi,s,z)$$

$$= \sum_{\nu=1}^{\infty} {}_or_{i\nu}(\xi)\psi_\nu(\xi,s,z)$$

where $\psi_i(\xi,s,z)$ for $i \geq 1$ is given in lemma 1.6, and ${}_or_{i\nu}(\cdot)$ is defined in (30).

Proof:

Upon taking transform in (39) we obtain:

$$\Phi_{ij}^{**}(\xi,s) = \sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} {}_oq_{i\nu}^{(n)}(\xi)\ \psi_{\nu j}^{**}(\xi,s)$$

$$= \sum_{\nu=1}^{\infty} {}_or_{i\nu}(\xi)\ \psi_{\nu j}^{**}(\xi,s)$$

Multiplying both sides by z^j and summing with respect to j we get (47).

Theorem 1.1

For $R(s) > 0$, $R(\xi) \geq 0$ and $|z| \leq 1$ the transform $\theta_i(\xi,s,z)$ of the joint distribution $\theta_{ij}(t,x)$ of queue length and virtual waiting time at time t for the tandem queue with non-zero switching rule is given by:

(48)

$$\theta_i(\xi,s,z) = \Phi_i(\xi,s,z) + \frac{\gamma^i(\xi)}{\xi+\lambda-\lambda\gamma(\xi)}\left[1+\lambda\ \Phi_1(\xi,s,z)\right],$$

where $\Phi_i(\xi,s,z)$ for $i \geq 1$ is given by lemma 1.7.

<u>Proof</u>:

The transform of (38) yields:

(49) $$\theta^{**}_{ij}(\xi,s) = \Phi^{**}_{ij}(\xi,s) + m_1(\xi)\Phi^{**}_{1j}(\xi,s)$$

$$+ \delta_{oj}\int_0^\infty e^{-\xi t}\ P\left\{\eta_1^{(k)}(t)=0\,|\,\xi^{(k)}(0)=i\right\} dt ,$$

The Kronecker delta in the last term is due to the fact that:

$$P\left\{\xi^{(k)}(t)=j,\eta_1^{(k)}(t)=0\,|\,\xi(0)=i\right\}$$

$$= 0 \quad \text{if } j \neq 0 ,$$

$$= P\left\{\eta_1^{(k)}(t)=0 \mid \xi(0)=i\right\} \text{ if } j=0 ,$$

If $M(\cdot)$ is the renewal function of the general renewal process formed by the ends of busy periods and $m(\xi)$ its L.S.T., then:

(50) $$\int_0^\infty e^{-\xi t}\ P\left\{\eta_1^{(k)}(t)=0 \mid \xi^{(k)}(0)=i\right\} dt$$

$$= \int_0^\infty e^{-\xi t}\int_0^t e^{-\lambda(t-u)}\ d\,M(u) ,$$

$$= \frac{m(\xi)}{\xi+\lambda} ,$$

where u is the end point of the last busy period before time t and no customer arrives between u and t.

Since the input is Poisson, between two successive busy periods there is a negative exponential idle period. Consider the renewal process formed by the end points of busy periods and let $F_1(\cdot)$ be the distribution function of the initial renewal and $F(\cdot)$ be the common distribution function of other renewals. Then:

$$F_1(x)=G^{(i)}(x), \text{ which is the i-fold convolution of } G(\cdot)$$

$$F(x) = \int_0^x [1-e^{-\lambda(x-u)}]\, d\, G(u) \ ,$$

where $G(\cdot)$ is the distribution function of busy periods.

Hence the renewal function $M(t)$, which is the expected number of renewals in $[0,t]$, is given by:

$$M(t) = (F_1 * \sum_{n=0}^{\infty} F^{(n)})(t) \ ,$$

Taking L.S.T. we get:

$$m(\xi) = f_1(\xi) \sum_{n=0}^{\infty} f^n(\xi)$$

$$= \frac{f_1(\xi)}{1-f(\xi)} \ , \qquad R(\xi) > 0 \ ,$$

where $f_1(\xi)$ and $f(\xi)$ are the L.S.T. of $F_1(\cdot)$ and $F(\cdot)$ respectively, which are given by $f_1(\xi) = \gamma^i(\xi)$ and

$$f(\xi) = \frac{\lambda\ \gamma(\xi)}{\lambda+\xi} \ .$$

That is:

$$m(\xi) = \frac{\gamma^{i}(\xi)}{1 - \frac{\lambda}{\lambda+\xi}\gamma(\xi)}$$

Substitution of this in (50) yields that:

$$(51)\qquad \int_0^\infty e^{-\xi t} P\left\{\eta_1^{(k)}(t)=0 \mid \xi^{(k)}(0)=i\right\} dt$$

$$= \frac{\gamma^{i}(\xi)}{\xi+\lambda-\lambda\gamma(\xi)} ,$$

Also,

$$(52)\qquad m_1(\xi) = \frac{\lambda}{\xi+\lambda} m(\xi)$$

$$= \frac{\lambda\ \gamma^{i}(\xi)}{\xi+\lambda-\lambda\gamma(\xi)} ,$$

The relation (52) between the L.S.T. of $M_1(t)$ and $M(t)$ holds because of the fact that the beginnings of busy periods are obtained by adding negative exponential idle periods to the end points of busy periods.

Substitution of (51) and (52) in (49) gives:

$$\theta_{ij}^{**}(\xi,s) = \Phi_{ij}^{**}(\xi,s) + \frac{\lambda\ \gamma^{i}(\xi)}{\xi+\lambda-\lambda\gamma(\xi)} \Phi_{1j}^{**}(\xi,s) + \delta_{oj} \frac{\gamma^{i}(\xi)}{\xi+\lambda-\lambda\gamma(\xi)}$$

Multiplying both sides by z^j and summing with respect to j we get the theorem.

5. Distribution of Virtual Waiting time

The stochastic behavior of the process $\{\eta_1^{(k)}(t),\ 0 \leq t < \infty\}$ is as follows: $\eta_1^{(k)}(0)$ is the initial occupation time of the server. If $\eta_1^{(k)}(0+) = 0$, then the server is idle at time $t = 0+$ and until the arrival of a customer who initiates a busy period.

Consider the arrival times $t_1, t_2, \ldots.$ within a busy period which started at $t = 0$ with $i > 0$ customers in unit 1. t_n is the n-th arrival point of the busy period. Let $i = mk + r$ $(0 < r < k$, m is a positive integer$)$ and $X_n^{(\nu)}$ be the service time of the n-th customer in unit ν, $\nu=1,2$. Then at t_n the arriving customer has a service time $X_{i+n}^{(1)}$ in unit 1. Hence at $t_n + 0$ the virtual customer has to wait a further $X_{i+n}^{(1)}$ units of time more to enter service in unit 1, provided i+n is not a multiple of k. That is, if i+n is not a multiple of k then at t_n $\eta_1^{(k)}(t)$ has a jump of magnitude $X_{i+n}^{(1)}$. On the other hand if i+n is a multiple of k then the virtual customer has to wait until the completion of that cycle and hence at t_n $\eta_1^{(k)}(t)$ has a jump of magnitude:

$$X_{i+n}^{(1)} + X_{i+n-k+1}^{(2)} + X_{i+n-k+2}^{(2)} + \ldots + X_{i+n}^{(2)}$$

This is shown in Figure 1.

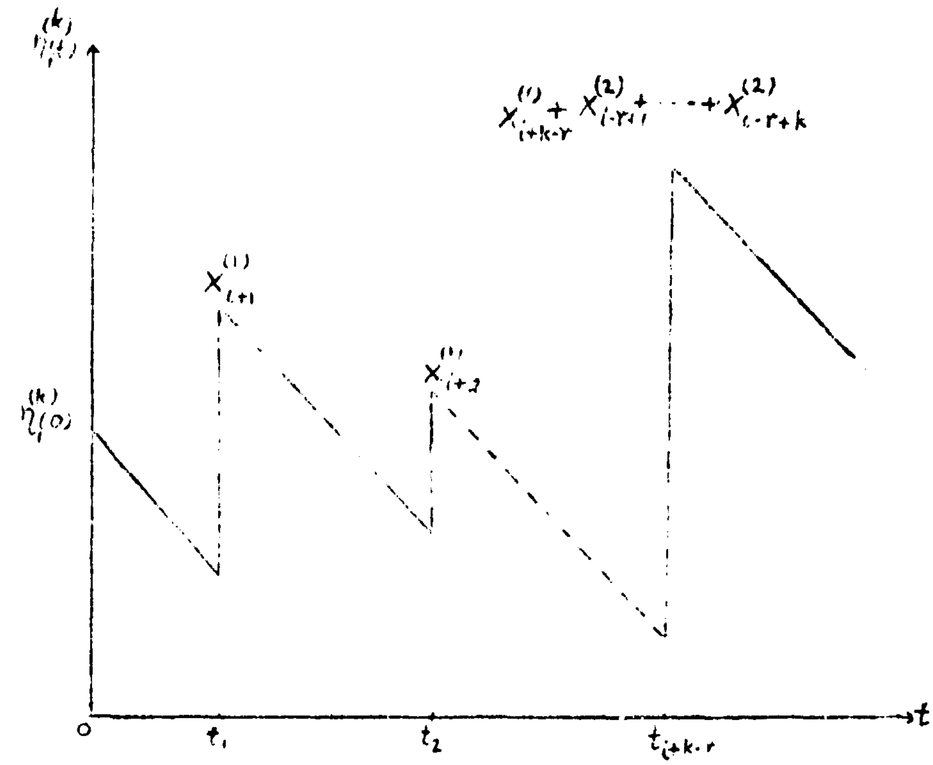

Figure 1

Graph of the Stochastic Behavior of the Process $\left\{\eta_1^{(k)}(t), 0 \leq t < \infty\right\}$

For $i \geq 1$, let ${}_1W_i(t,x) = P\left\{\eta_1^{(k)}(t) \leq x \mid \xi^{(k)}(0)=i\right\}$ be the distribution function of virtual waiting time $\eta_1^{(k)}(t)$, given that at t=0 there were i customers. ${}_1W_i^*(t,s)$ is the L.S.T. of ${}_1W_i(t,x)$ and:

$$(53) \qquad {}_1W_i^{**}(\xi,s) = \int_0^\infty e^{-\xi t}\, {}_1W_i^*(t,s)\, dt$$

<u>Theorem 1.2</u>

For $R(s) > 0$ and $R(\xi) \geq 0$, the transform ${}_1W_i^{**}(\xi,s)$ of the distribution function ${}_1W_i(t,x)$ of the virtual waiting time $\eta_1^{(k)}(t)$ is given by:

(54) $$ {}_1W_i^{**}(\xi,s) = {}_1\Lambda_i^{**}(\xi,s) + \frac{\gamma^i(\xi)}{\xi+\lambda-\lambda\gamma(\xi)}\left[1+\lambda\ {}_1\Lambda_1^{**}(\xi,s)\right], $$

$$ i \geq 1 , $$

where ${}_1\Lambda_i^{**}(\xi,s)$ is given by:

(55)

$$ {}_1\Lambda_i^{**}(\xi,s) = \frac{1}{k}\sum_{m=0}^{k-1}\Big[h_2^{k-1}(s)[\omega_m h_2(s)-1][\xi+\lambda-s-\lambda\omega_m h_1(s)h_2(s)]\Big]^{-1} $$

$$ \cdot\Big[\omega_m[h_2^k(s)-1]\Big\{[\omega_m h_1(s)h_2(s)]^i - \gamma^i(\xi) $$

$$ + \sum_{j=1}^{k-1} {}_0r_{ij}(\xi)\sum_{\nu=j}^{k-1}(1-\omega_m^\nu)h_2^\nu(s)g_j^{(\nu)}(\xi,0)\Big\}\Big] $$

Proof:

From Theorem 1.1 the transform of the joint distribution $\theta_{ij}(t,x)$ of queue length and virtual waiting time is given by $\theta_i(\xi,s,z)$. Hence the transform of the distribution of virtual waiting time is obtained by taking the limit $z \to 1$ in $\theta_i(\xi,s,z)$. That is, from (48):

(56) $$ {}_1W_i^{**}(\xi,s) = \lim_{z\to 1}\ \theta_i(\xi,s,z) $$

$$ = \Phi_i(\xi,s,1) + \frac{\gamma^i(\xi)}{\xi+\lambda-\lambda\gamma(\xi)}\left[1+\lambda\Phi_1(\xi,s,1)\right] $$

Hence it suffices to prove that:

(57) $$ \theta_i(\xi,s,1) = {}_1\Lambda_i^{**}(\xi,s) $$

Lemma 1.7 gives:

$$\Phi_i(\xi,s,1) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} {}_oq_{ij}^{(n)}(\xi)\ \psi_j(\xi,s,1) \tag{58}$$

where from lemma 1.6 $\psi_j(\xi,s,1)$ is given by:

$$\psi_j(\xi,s,1) = \frac{1}{k}\sum_{m=0}^{k-1}\left[h_2^{k-1}(s)[\omega_m h_2(s)-1][\xi+\lambda-s-\lambda\omega_m h_1(s)h_2(s)]\right]^{-1}$$

$$\left[\omega_m[h_2^k(s)-1]\left\{[\omega_m h_1(s)h_2(s)]^j - q_j[\xi,\omega_m h_1(s)h_2(s)]\right.\right.$$

$$\left.\left. + \sum_{\nu=1}^{k-1}(1-\omega_m^\nu)h_2^\nu(s)g_j^{(\nu)}(\xi,0)\right\}\right]$$

Substitution of this in (58) leads to:

$$\Phi_i(\xi,s,1) = \frac{1}{k}\sum_{m=0}^{k-1}\left[h_2^{k-1}(s)[\omega_m h_2(s)-1][\xi+\lambda-s-\lambda\omega_m h_1(s)h_2(s)]\right]^{-1}$$

$$\left[\omega_m[h_2^k(s)-1]\sum_{n=0}^{\infty}\left\{{}_oq_i^{(n)}[\xi,\omega_m h_1(s)h_2(s)] - {}_oq_i^{(n)}(\xi,0)\right.\right.$$

$$- {}_oq_i^{(n+1)}[\xi,\omega_m h_1(s)h_2(s)]$$

$$\left.\left. + \sum_{j=1}^{\infty} {}_oq_{ij}^{(n)}(\xi)\sum_{\nu=1}^{k-1}(1-\omega_m^\nu)\ h_2^\nu(s)\ g_j^{(\nu)}(\xi,0)\right\}\right]$$

which establishes (57), noting that from (33):
${}_oq_{ij}^{(o)}(\xi) = \delta_{ij}$ and from (35): $\sum_{n=0}^{\infty} {}_oq_i^{(n)}(\xi,0) = \gamma^i(\xi)$.

Limiting Behavior of the Virtual Waiting time Process

The limiting behavior of the distribution of virtual waiting time is given by:

Theorem 1.3

If $\lambda\alpha_1 + \lambda\alpha_2 < 1$ the limiting distribution $\lim_{t\to\infty} {}_1W_i(t,x) = {}_1W(x)$ exists. The L.S.T. of ${}_1W(x)$ is given by:

$$(59) \qquad {}_1\omega(s) = (1-\lambda\alpha_1-\lambda\alpha_2)[1+\lambda\, {}_1\Lambda_1^{**}(0,s)]$$

where ${}_1\Lambda_1^{**}(0,s)$ is given by (55).

If $\lambda\alpha_1 + \lambda\alpha_2 \geq 1$, then $\lim_{t\to\infty} {}_1W_i(t,x) = 0$ for all x.

In order to prove this theorem we first show that:

Lemma 1.8

If $P_{io}(t) = P\{\eta_1^{(k)}(t) = 0 \mid \xi^{(k)}(0) = i\}$

then the limit $\lim_{t\to\infty} P_{io}(t) = P_o^*$ always exists. We have:

$$(60) \qquad P_o^* = \begin{cases} 1 - \lambda\alpha_1 - \lambda\alpha_2 & \text{if } \lambda\alpha_1 + \lambda\alpha_2 < 1 \\ 0 & \text{if } \lambda\alpha_1 + \lambda\alpha_2 \geq 1 \end{cases}$$

Proof:

If $M(\cdot)$ is the renewal function of the general renewal process formed by the ends of busy periods, then:

$$P_{io}(t) \equiv P\{\eta_1^{(k)}(t) = 0 \mid \xi^{(k)}(0) = i\} = \int_0^t e^{-\lambda(t-u)}\, dM(u)$$

By Smith's Key Renewal Theorem,

$$\lim_{t \to \infty} P_{io}(t) = \frac{1}{\mu} \int_0^\infty e^{-\lambda u}\, du \qquad \text{if } \lambda\alpha_1 + \lambda\alpha_2 < 1$$

$$= 0 \qquad \text{if } \lambda\alpha_1 + \lambda\alpha_2 \geq 1$$

where $\mu = \frac{1}{\lambda(1-\lambda\alpha_1-\lambda\alpha_2)}$ is the mean renewal time. Hence the lemma is proved.

<u>Proof of Theorem 1.3</u>:

Summing equation (38) with respect to j we get the distribution of virtual waiting time as:

(61) $$_1W_i(t,x) = {}_1\Lambda_i(t,x) + \int_0^t {}_1\Lambda_1(t-u,x)\, dM_1(u)$$

$$+ P\left\{\eta_1^{(k)}(t)=0 \mid \xi^{(k)}(0)=i\right\} \; U(x)$$

where:

(62) $$_1\Lambda_i(t,x) = \sum_{j=0}^{\infty} \Phi_{ij}(t,x)$$

Taking L.S.T. of (61):

(63)

$$_1W_i^*(t,s) = {}_1\Lambda_i^*(t,s) + \int_0^t {}_1\Lambda_1^*(t-u,s)\, dM_1(u)$$

$$+ P\left\{\eta_1^{(k)}(t)=0 \mid \xi^{(k)}(0) = i\right\}$$

Using Smith's Key Renewal Theorem (Theorem 4, Appendix D) and lemma 1.8 and taking the limit of (63) we have:

(64)

$$\lim_{t \to \infty} {}_1W_i^*(t,s) = \lambda(1-\lambda\alpha_1-\lambda\alpha_2)\int_0^\infty {}_1\Lambda_1^*(u,s)du+(1-\lambda\alpha_1-\lambda\alpha_2) ,$$

$$\text{if } 1-\lambda\alpha_1 - \lambda\alpha_2 > 0$$

$$= 0 \quad \text{if } 1-\lambda\alpha_1 - \lambda\alpha_2 \leq 0.$$

since from (41) it can be shown that $\lim_{t \to \infty} \sum_{j=0}^{\infty} \psi_{ij}^*(t,s) = 0$

which implies from (39) that $\lim_{t \to \infty} {}_1\Lambda_i^*(t,s) = \lim_{t \to \infty} \sum_{j=0}^{\infty} \Psi_{ij}^*(t,s)=0.$

Again, (64) can be written as:

(65)

$$\lim_{t \to \infty} {}_1W_i^*(t,s) = (1-\lambda\alpha_1-\lambda\alpha_2)[1+\lambda\cdot{}_1\Lambda_1^{**}(0,s)] \text{ if } 1-\lambda\alpha_1-\lambda\alpha_2 > 0,$$
$$= 0 \qquad \text{if } 1-\lambda\alpha_1-\lambda\alpha_2 \leq 0$$

From (55) it can be shown that ${}_1\Lambda_1^{**}(0,s)$ is continuous at s=0.

Hence by Zygmund's Theorem (Theorem 1, Appendix D) the limiting distribution $\lim_{t \to \infty} {}_1W(t,x) = {}_1W(x)$ exists and the L.S.T. of ${}_1W(x)$ is given by (65).

Formula (55) can be rewritten as:

(66)

$$
{}_1\Lambda_1^{**}(0,s) = \Bigg(\frac{[h_2^k(s)-1][h_1(s)h_2(s)-1]}{kh_2^{k-1}(s)[h_2(s)-1][\lambda-s-\lambda h_1(s)h_2(s)]}\Bigg)
$$

$$
+ \Bigg(\frac{1}{k}\sum_{m=1}^{k-1}\Big[h_2^{k-1}(s)[\omega_m h_2(s)-1][\lambda-s-\lambda\omega_m h_1(s)h_2(s)]\Big]^{-1}
$$

$$
\Big[\omega_m[h_2^k(s)-1]\Big\{\omega_m h_1(s)h_2(s)-1
$$

$$
+ \sum_{j=1}^{k-1} {}_0r_{1j}(0) \sum_{\nu=j}^{k-1} (1-\omega_m^{\nu})h_2^{\nu}(s)g_j^{(\nu)}(0,0)\Big\}\Big]\Bigg)
$$

$$
= \zeta_1(s) + \zeta_2(s)
$$

where $\zeta_1(s)$ and $\zeta_2(s)$ are respectively the first and the second term of (66)

Taking the limit $s \to 0$ in (66), we see that the numerator of $\zeta_2(s)$ is zero while its denominator is non-zero. Hence:

$$
{}_1\Lambda_1^{**}(0,0) = \zeta_1(0+)
$$

$$
= \frac{\alpha_1 + \alpha_2}{1-\lambda\alpha_1 - \lambda\alpha_2} \quad ,
$$

which together with (59) gives ${}_1\omega(0+) = 1$.

Expected Value of the Limiting Distribution of Virtual Waiting time

Let $M_{\eta_1}(k)$ denote the expected value of the limiting distribution of $\eta_1^{(k)}(t)$. Then

$$(67) \qquad M_{\eta_1}(k) = - \frac{\partial}{\partial s}\, {}_1\omega(s)\Big]_{s=0}$$

$$= - \lambda(1-\lambda\alpha_1-\lambda\alpha_2)\, \frac{\partial}{\partial s}\, {}_1\Lambda_1^{**}(0,s)\Big]_{s=0}$$

From (66):

$$(68) \qquad \frac{\partial}{\partial s}\, {}_1\Lambda_1^{**}(0,s)\Big]_{s=0+} = \nu_1'(0+) + \nu_2'(0+)$$

where the number of primes indicates the number of successive derivatives taken with respect to s. Let $\Delta_1(s)$ denote the numerator and $\Delta_2(s)$ the denominator of $\nu_1(s)$, so that:

$$(69) \qquad \nu_1'(0+) = \frac{\Delta_2(s)\Delta_1'(s) - \Delta_2'(s)\Delta_1(s)}{[\Delta_2(s)]^2}\Bigg]_{s=0+}$$

Applying de l'Hopital's rule four times on the right hand side of (69) we get:

$$\nu_1'(0+) = \Big[\Delta_2''(0)\Delta_1'''(0) - \Delta_1''(0)\Delta_2'''(0)\Big]\ \Big[\Delta_2''(0)\Big]^2$$

Where after simplifying we obtain:

$$\Delta_1''(0) = 2k\alpha_2(\alpha_1 + \alpha_2)$$

$$\Delta_1'''(0) = -3k\left[\alpha_2(\beta_1+2\alpha_1\alpha_2+\beta_2)+(k-1)\alpha_2^2(\alpha_1+\alpha_2)+\beta_2(\alpha_1+\alpha_2)\right]$$

$$\Delta_2''(0) = -2k\alpha_2(-1+\lambda\alpha_1+\lambda\alpha_2)$$

$$\Delta_2'''(0) = 3k\Big[2(k-1)\alpha_2^2(-1+\lambda\alpha_1+\lambda\alpha_2)+3\lambda\alpha_2(\beta_1+2\alpha_1\alpha_2+\beta_2)$$

$$+ 3\beta_2(-1+\lambda\alpha_1+\lambda\alpha_2)\Big]$$

Hence:

$$\zeta_1'(0+) = \frac{(K-1)\alpha_2(\alpha_1+\alpha_2)(1-\lambda\alpha_1-\lambda\alpha_2)-(\beta_1+2\alpha_1\alpha_2+\beta_2)}{2(1-\lambda\alpha_1-\lambda\alpha_2)^2} \tag{70}$$

Differentiating $\zeta_2(s)$ with respect to s and setting $s=0$:

$$\zeta_2'(0) = \frac{\alpha_2}{\lambda}\sum_{m=1}^{k-1}\frac{\omega_m}{(1-\omega_m)^2}\Big\{\omega_m-1$$

$$+\sum_{j=1}^{k-1}{}_0r_{1j}(0)\sum_{\nu=j}^{k-1}(1-\omega_m^{\nu})\,g_j^{(\nu)}(0,0)\Big\},$$

$$= \frac{-\alpha_2}{\lambda}\sum_{m=1}^{k-1}\frac{\omega_m}{1-\omega_m} \tag{71}$$

$$+\frac{\alpha_2}{\lambda}\sum_{j=1}^{k-1}{}_0r_{1j}(0)\sum_{\nu=j}^{k-1}\sum_{m=1}^{k-1}\frac{\omega_m(1-\omega_m^{\nu})}{(1-\omega_m)^2}\,g_j^{(\nu)}(0,0),$$

From the properties of the roots of the equation $z^k-1 = 0$ we find that:

$$\sum_{m=1}^{k-1} \frac{1}{1-\omega_m} = \frac{k-1}{2} \quad ,$$

$$(72) \qquad \sum_{m=1}^{k-1} \frac{\omega_m}{1-\omega_m} = \sum_{m=1}^{k-1} \left(-1 + \frac{1}{1-\omega_m}\right) = -\frac{k-1}{2}$$

By the method of partial fractions we get:

$$\omega_m^{\nu+1} \equiv 1-(\nu+1)(1-\omega_m)+\nu(1-\omega_m)^2+(\nu-1)\omega_m(1-\omega_m)^2+(\nu-2)\omega_m^2(1-\omega_m)^2$$

$$+ \cdots\cdots + 2\omega_m^{\nu-2}(1-\omega_m)^2 + \omega_m^{\nu-1}(1-\omega_m)^2$$

which gives:

$$\sum_{m=1}^{k-1} \frac{\omega_m(1-\omega_m^{\nu})}{(1-\omega_m)^2} = -\sum_{m=1}^{k-1} \frac{1}{1-\omega_m} + \sum_{m=1}^{k-1} \frac{1}{(1-\omega_m)^2} - \sum_{m=1}^{k-1} \frac{\omega_m^{\nu+1}}{(1-\omega_m)^2}$$

$$= \sum_{m=1}^{k-1} \left[\frac{\nu}{1-\omega_m} - \nu-(\nu-1)\omega_m-(\nu-2)\omega_m^2 - \cdots\cdots \right.$$
$$\left. \cdots\cdots - 2\omega_m^{\nu-2} - \omega_m^{\nu-1}\right]$$

$$= \nu\left(\frac{k-1}{2}\right)-\nu(k-1)+[(\nu-1)+(\nu-2)+ \cdots + 2+1]$$

since $\sum_{m=1}^{k-1} \omega_m^r = -1$ for $1 \leq r \leq k-1$.

That is:

$$(73) \qquad \sum_{m=1}^{k-1} \frac{\omega_m(1-\omega_m^{\nu})}{(1-\omega_m)^2} = -\frac{\nu(k-\nu)}{2}$$

Substitution of (72) and (73) in (71) yields:

(74) $$\eta_2'(0) = \frac{(K-1)\,\alpha_2}{2\,\lambda}$$

$$- \frac{\alpha_2}{2\lambda} \sum_{j=1}^{k-1} {}_o r_{1j}(0) \sum_{\nu=j}^{k-1} \nu(k-\nu)\, g_j^{(\nu)}(0,0)$$

Formula (68) together with (70) and (74) leads to:

(75)

$$\frac{\partial}{\partial s}\, {}_1\Lambda_1^{**}(0,s) = \frac{1}{2(1-\lambda\alpha_1-\lambda\alpha_2)^2}\Big[(k-1)\alpha_2(\alpha_1+\alpha_2)(1-\lambda\alpha_1-\lambda\alpha_2)$$

$$- (\beta_1+2\alpha_1\alpha_2+\beta_2)\Big] + \frac{(K-1)\alpha_2}{2\lambda}$$

$$- \frac{\alpha_2}{2\lambda} \sum_{j=1}^{k-1} {}_o r_{1j}(0) \sum_{\nu=j}^{k-1} \nu(k-\nu) g_j^{(\nu)}(0,0)$$

It follows from (67) that:

(76) $$M_{\eta_1}(k) = \frac{\lambda(\beta_1+2\alpha_1\alpha_2+\beta_2)}{2(1-\lambda\alpha_1-\lambda\alpha_2)} - \frac{(K-1)\alpha_2}{2}$$

$$+ \frac{\alpha_2}{2}(1-\lambda\alpha_1-\lambda\alpha_2) \sum_{j=1}^{k-1} {}_o r_{1j}(0) \sum_{\nu=j}^{k-1} \nu(k-\nu) g_j^{(\nu)}(0,0)$$

where ${}_o r_{1j}(\cdot)$ is defined in (30).

It is worth noting that for k=1:

$$M_{\eta_1}(1) = \lambda(\beta_1+2\alpha_1\alpha_2+\beta_2) \mid 2(1-\lambda\alpha_1-\lambda\alpha_2)$$

which is the expected value of the limiting distribution of the virtual waiting time of an M|G|1 queue with impute rate λ and service time distribution $H_1 * H_2(\cdot)$. For k=1 the tandem model

reduces to an $M|G|1$ queue with service time distribution $H_1 * H_2(\cdot)$.

Conjecture:

For all $k \geq 1$:

(77) $$M_{\eta_1}(k+1) < M_{\eta_1}(k)$$

The proof for k=1 is simple:

From (76) we have:

$$M_{\eta_1}(2) - M_{\eta_1}(1) = \frac{\alpha}{2}\left[(1-\lambda\alpha_1-\lambda\alpha_2)_o r_{11}(0) g_1^{(1)}(0,0)-1\right]$$

$$< \frac{\alpha}{2}\left[(1-\lambda\alpha_1-\lambda\alpha_2)_o r_{11}(0)-1\right] < 0$$

since from Theorem 3 of Appendix B $\sum_{j=1}^{\infty} {}_o r_{1j}(0) < \frac{1}{1-\lambda\alpha_1-\lambda\alpha_2}$

6. Queue length Process

Let $P_{ij}(t)$ denote the probability that at t, j customers are in unit 1, given that the service started at time t=0 with i customers. That is:

(78) $$P_{ij}(t) = P\left\{\xi^{(k)}(t) = j \mid \xi^{(k)}(0) = i\right\}$$

Let $\pi_{ij}(\xi)$ be the Laplace transform of $P_{ij}(t)$ and:

(79) $$\pi_i(\xi,z) = \sum_{j=o}^{\infty} \pi_{ij}(\xi) z^j \ , \ |z| < 1, \ R(\xi) \geq 0$$
$$\text{or } |z| \leq 1, \ R(\xi) > 0$$

For i=o and $j \geq 0$ we have:

(80) $$\pi_{oj}(\xi) = \frac{\delta_{oj}}{\lambda+\xi} + \frac{\lambda}{\lambda+\xi}\pi_{1j}(\xi) \quad , \text{ and}$$

(81) $$\pi_o(\xi,z) = \frac{1}{\lambda+\xi}\left[1 + \lambda\,\pi_1(\xi,z)\right]$$

The equation (80) is obtained by considering:

(i) If $j=0$ then there can be either no arrival in $[0,t]$ or there is a negative exponential idle period followed by a busy period.

(ii) If $j > 0$ there is a negative exponential idle period followed by a busy period, and in this case the first term on the right hand side of (80) vanishes.

For $i > 0$ we have:

<u>Theorem 1.4</u>:

The generating function $\pi_i(\xi,z)$ is given by:

(82) $$\pi_i(\xi,z) = \chi_i(\xi,z) + \frac{\gamma^i(\xi)}{\xi+\lambda-\lambda\gamma(\xi)}\left[1 + \lambda\,\chi_1(\xi,z)\right]$$

where

(83) $$\chi_i(\xi,z) = \Big[(\xi+\lambda-\lambda z)[z-h_1(\xi+\lambda-\lambda z)]\Big]^{-1}$$

$$\Big[z^i[z-h_1(\xi+\lambda-\lambda z)]-z[1-h_1(\xi+\lambda-\lambda z)]\,\gamma^i(\xi)$$

$$-(z-1)h_1(\xi+\lambda-\lambda z)\Big\{{}_or_i(\xi,z)-\sum_{j=1}^{\infty}{}_or_{ij}(\xi)[g_j^{(k)}(\xi,z)+\sum_{\nu=1}^{k-1}g_j^{(\nu)}(\xi,0)]\Big\}\Big],$$

and ${}_or_{ij}(\xi)$, ${}_or_i(\xi,z)$ are defined in (30) and (31).

Proof:

The generating function $\pi_i(\xi,z)$ is obtained by taking the limit as $s \to 0+$ in the transform $\theta_i(\xi,s,z)$ of the joint distribution $\theta_{ij}(t,x)$ of queue length and virtual waiting time.

From Theorem 1.1 we have:

$$\pi_i(\xi,z) = \lim_{s \to 0+} \theta_i(\xi,s,z)$$

(84) $$= \Phi_i(\xi,0,z) + \frac{\gamma^i(\xi)}{\xi+\lambda-\lambda\gamma(\xi)} \left[1+\lambda\, \Phi_1(\xi,0,z)\right]$$

Hence it suffices to prove that:

(85) $$\Phi_i(\xi,0,z) = \chi_i(\xi,z)$$

From lemma 1.7 $\Phi_i(\cdot,\cdot,\cdot)$ is given by:

(86) $$\Phi_i(\xi,0,z) = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} {}_0q_{ij}^{(n)}(\xi)\, \psi_j(\xi,0,z)$$

where $\psi_j(\xi,0,z)$ from lemma 1.6 is given by:

(87) $$\psi_j(\xi,0,z) = \Big[(\xi+\lambda-\lambda z)[z-h_1(\xi+\lambda-\lambda z)]\Big]^{-1}$$

$$\Big[[1-h_1(\xi+\lambda-\lambda z)]z^{j+1} - [z-h_1(\xi+\lambda-\lambda z)]q_j(\xi,z)$$

$$+ (z-1)h_1(\xi+\lambda-\lambda z)\Big\{g_j^{(k)}(\xi,z) + \sum_{\nu=1}^{k-1} g_j^{(\nu)}(\xi,0)\Big\}\Big]$$

since other terms in the summation on the right hand side of (40) for m=1,2,....,k-1 vanish as $s \to o$. Substituting (87) in (86) we find that:

$$(88)\qquad \phi_i(\xi,0,z) = \Big[(\xi+\lambda-\lambda z)[z-h_1(\xi+\lambda-\lambda z)]\Big]^{-1}$$

$$\Big[z[1-h_1(\xi+\lambda-\lambda z)] \sum_{n=0}^{\infty} [{}_0q_i^{(n)}(\xi,z)-{}_0q_i^{(n)}(\xi,0)]$$

$$- [z-h_1(\xi+\lambda-\lambda z)] \sum_{n=0}^{\infty} {}_0q_i^{(n+1)}(\xi,z)$$

$$+ (z-1)h_1(\xi+\lambda-\lambda z) \sum_{j=1}^{\infty} {}_0r_{ij}(\xi)\Big\{g_j^{(k)}(\xi,z)+\sum_{\nu=j}^{k-1} g_j^{(\nu)}(\xi,0)\Big\}\Big]$$

$$= \Big[(\xi+\lambda-\lambda z)[z-h_1(\xi+\lambda-\lambda z)]\Big]^{-1}$$

$$\Big[z^i[z-h_1(\xi+\lambda-\lambda z)]-z[1-h_1(\xi+\lambda-\lambda z)]\ \gamma^i(\xi)$$

$$- (z-1)h_1(\xi+\lambda-\lambda z)\ \Big\{{}_0r_i(\xi,z)-\sum_{j=1}^{\infty} {}_0r_{ij}(\xi)\ [g_j^{(k)}(\xi,z)$$

$$+ \sum_{\nu=1}^{k-1} g_j^{(\nu)}(\xi,0)]\Big\}\Big]$$

which proves (85).

Limiting Behavior of Queue length

The limit of $P_{ij}(t)$ as $t \to \infty$ always exists by a theorem of Smith (1955).

Let us denote:

$$(89)\qquad P_j^* = \lim_{t\to\infty} P_{ij}(t)$$

and

$$(90)\qquad P^*(z) = \sum_{j=0}^{\infty} P_j^* z^j\ ,\quad |z|<1,$$

Then by a standard Tauberian Theorem (Theorem 5, Appendix D) we have:

$$(91) \qquad P_j^* \equiv \lim_{t \to \infty} \frac{1}{t} \int_0^t P_{ij}(t)\, dt$$

$$= \lim_{\xi \to 0} \xi \int_0^\infty e^{-\xi t} P_{ij}(t)\, dt$$

which together with Theorem 1.4 gives:

$$P^*(z) = \lim_{\xi \to 0} \xi\, \pi_i(\xi, z)$$

$$= \lim_{\xi \to 0} \frac{\xi \gamma^i(\xi)}{\xi+\lambda-\lambda\gamma(\xi)} [1+\lambda\, X_1(\xi,z)]$$

$$(92) \qquad = \begin{cases} (1-\lambda\alpha_1-\lambda\alpha_2)[1+\lambda X_1(0,z)] & \text{if } 1-\lambda\alpha_1-\lambda\alpha_2 > 0 \\ 0 & \text{if } 1-\lambda\alpha_1-\lambda\alpha_2 \leq 0 \end{cases}$$

where $X_1(0,z)$ from (83) is given by:

$$(93) \qquad X_1(0,z) = \frac{1}{\lambda[z-h_1(\lambda-\lambda z)]} \Big\{ -z+h_1(\lambda-\lambda z)\Big[{}_0r_1(0,z)$$

$$- \sum_{j=1}^{\infty} {}_0r_{1j}(0)[g_j^{(k)}(0,z)+\sum_{\nu=1}^{k-1} g_j^{(\nu)}(0,0)]\Big] \Big\}$$

Substitution of (93) in (92) yields the following:

<u>Theorem 1.5</u>

The generating function $P^*(z)$ of the limiting probabilities P_j^* of $P_{ij}(t)$ is given by:

(94)

$$P^*(z) = (1-\lambda\alpha_1-\lambda\alpha_2)h_1(\lambda-\lambda z)[z-h_1(\lambda-\lambda z)]^{-1}\Big\{-1 + {}_o r_1(0,z)$$

$$- \sum_{j=1}^{\infty} {}_o r_{1j}(0)\Big[g_j^{(k)}(0,z)+\sum_{\nu=1}^{k-1} g_j^{(\nu)}(0,0)\Big]\Big\}$$

where ${}_o r_{1j}(\cdot)$ is defined by (30).

The Steady State Expected Queue length

Let $M_\xi(k)$ denote the mean of the limiting distribution of $\xi^{(k)}(t)$ as $t \to \infty$. Theorem 1.5 gives:

$$M_\xi(k) = \frac{\partial}{\partial z} P^*(z)\Big]_{z=1}$$

(95)

$$= \frac{(1-\lambda\alpha_1-\lambda\alpha_2)}{2(1-\lambda\alpha_1)^2}\Big[[2\lambda\alpha_1(1-\lambda\alpha_1)+\lambda^2\beta_1]\sum_{j=o}^{\infty} j\Big\{{}_o r_{1j}(0)$$

$$- \sum_{\nu=1}^{\infty} {}_o r_{1\nu}(0)\, g_{\nu j}^{(k)}(0)\Big\}$$

$$+ (1-\lambda\alpha_1)\sum_{j=o}^{\infty} j(j-1)\Big\{{}_o r_{1j}(0)-\sum_{\nu=1}^{\infty} {}_o r_{1\nu}(0) g_{\nu j}^{(k)}(0)\Big\}\Big]$$

$$= \frac{(1-\lambda\alpha_1-\lambda\alpha_2)}{2(1-\lambda\alpha_1)^2}\sum_{j=1}^{\infty}\Big[j\Big\{[2\lambda\alpha_1(1-\lambda\alpha_1)+\lambda^2\beta_1]+(j-1)(1-\lambda\alpha_1)\Big\}$$

$$\Big\{{}_o r_{1j}(0) - \sum_{\nu=1}^{\infty} {}_o r_{1\nu}(0)\, g_{\nu j}^{(k)}(0)\Big\}\Big]$$

CHAPTER II

SINGLE SERVER TANDEM QUEUE WITH ZERO SWITCHING

Here we consider a tandem queue with a zero switching rule in units 1 and 2. At t=0 the server starts in unit 1 and continues to serve there until the queue in unit 1 becomes empty. After completing the 1-task the server switches to unit 2, serves all the customers there, then switches back to unit 1 and continues in this manner. If the whole system is empty the server waits in unit 1 for the arrival of a customer who initiates a busy period.

The analysis of the queue with zero switching case is easier than non-zero switching case. The distribution of busy period for the zero switching case is the same as in non-zero switching case (Chap. I section 2).

1. Transition Probabilities of the Basic Imbedded Semi-Markov Process

We use the same notations as in Chap. I.

$$(1) \quad Q_{ij}(x) = P\left\{\xi_n = j,\ T_n \le x \mid \xi_{n-1} = i\right\}$$

$$= \sum_{r=i}^{\infty} \int_{\substack{o\\(u)}}^{x} \int_{\substack{u\\(v)}}^{x} dG_{io}^{(r)}(u) e^{-\lambda(v-u)} \frac{[\lambda(v-u)]^j}{j!} dH_2^{(r)}(v-u)$$

The transforms $q_{ij}(s)$ and $q_i(s,z)$ of $Q_{ij}(\cdot)$ are given by:

$$(2) \qquad q_{ij}(s) = \sum_{r=i}^{\infty} g_{io}^{(r)}(s) \int_o^{\infty} e^{-(s+\lambda)x} \frac{(\lambda x)^j}{j!} dH_2^{(r)}(x)$$

$$q_i(s,z) = \sum_{r=i}^{\infty} g_i^{(r)}(s,0)\ h_2^r(s+\lambda-\lambda z)$$

$$(3) \qquad = \gamma_1^i\left\{s,\ h_2(s + \lambda - \lambda z)\right\}$$

(by lemma 1.2)

If ${}_oQ_{ij}^{(n)}(\cdot)$ are the taboo probabilities defined in (1.28) then:

$${}_oQ_{ij}^{(o)}(x) = \delta_{ij}\ U(x)$$

and

$$(4) \qquad {}_oQ_{ij}^{(n)}(x) = \sum_{\nu=1}^{\infty} \int_o^x {}_oQ_{i\nu}^{(n-1)}(x-u) dQ_{\nu j}(u), \qquad n \ge 1$$

Their transforms are given by:

$$(5) \qquad \begin{aligned} {}_oq_{ij}^{(o)}(s) &= \delta_{ij} \\ {}_oq_{ij}^{(n)}(s) &= \sum_{\nu=1}^{\infty} {}_oq_{i\nu}^{(n-1)}(s)\ q_{\nu j}(s) \quad , \ n \ge 1\ , \end{aligned}$$

$$
{}_oq_i^{(0)}(s,z) = z^i
$$

(6)

$$
{}_oq_i^{(n)}(s,z) = \sum_{\nu=1}^{\infty} {}_oq_{i\nu}^{(n-1)}(s)\; q_\nu(s,z),\; n \geq 1,
$$

A further discussion of these transition probabilities is given in Appendix B.

2. The Joint Distribution of Queue length and Virtual Waiting time

For the definitions of queue length and virtual waiting time we refer to Chap. I sec. 4. Let $\xi(t)$ and $\eta_1(t)$ respectively denote the queue length and virtual waiting time at t. Define:

(7) $$\theta_{ij}(t,x) = P\{\xi(t)=j,\; \eta_1(t) \leq x \mid \xi(0)=i\}$$

(8)

$$
\Phi_{ij}(t,x) = P\{\xi(t)=j,\; 0 < \eta_1(t) \leq x, \eta_1(\tau)\neq 0 \text{ for all } \tau \in (0,t] \mid \xi(0) = i\}
$$

Analogous to equation (1.38) we obtain:

(9) $$\theta_{ij}(t,x) = \Phi_{ij}(t,x) + \int_0^t \Phi_{1j}(t-u,x)\; dM_1(u) + P\{\xi(t)=j,\; \eta_1(t)=0 \mid \xi(0)=i\}\; U(x)\;,$$

Let $\psi_{ij}(t,x)$ denote the probability that at time t the original cycle has not ended that $\zeta(t) = j$, $0 < \eta_1(t) \leq x$ and $\eta_1(\tau) \neq 0$ for all $\tau \in (0,t]$, given that at t=0 the service started in unit 1 with i customers.

We define the transforms $\theta_{ij}^{**}(\xi,s)$, $\Phi_{ij}^{**}(\xi,s)$, $\psi_{ij}^{**}(\xi,s)$, $\theta_i(\xi,s,z)$, $\Phi_i(\xi,s,z)$ and $\psi_i(\xi,s,z)$ as in Chap. I sec. 4.

Lemma 2.1

For $R(s) > 0$, $R(\xi) \geq 0$ and $|z| \leq 1$, the transform $\psi_i(\xi,s,z)$ is given by:

(10)
$$\psi_i(\xi,s,z) = \frac{1}{\xi+\lambda-s-\lambda z h_1(s)} \Big\{ q_i(\xi,\frac{\xi+\lambda-s}{\lambda}) - q_i(\xi, z h_1(s))$$
$$+ z\left[(zh_1(s))^i - \gamma_1^i(\xi)\right]\left[h_1(s) - h_1(\xi+\lambda-\lambda z h_1(s))\right]$$
$$\left[z h_1(s) - h_1(\xi+\lambda-\lambda z h_1(s))\right]^{-1}\Big\} \quad , \ i \geq 1 \ ,$$

Proof:

The probability $\psi_{ij}(t,x)$ is given by:

(11)
$$\psi_{ij}(t,x) = \sum_{r=i}^{\infty} \underset{(u)}{\int_0^t} \underset{(v)}{\int_t^{t+x}} \underset{(v_1)}{\int_v^{t+x}} dG_{io}^{(r)}(u) e^{-\lambda(t-u)} \frac{[\lambda(t-u)]^j}{j!} dH_2^{(r)}(v-u) dH_1^{(j)}(v_1-v)$$
$$+ \sum_{r=0}^{\infty} \sum_{\nu=1}^{j} \underset{(u)}{\int_0^t} \underset{(v)}{\int_t^{t+x}} \underset{(v_1)}{\int_v^{t+x}} dG_{i\nu}^{(r)}(u)\, e^{-\lambda(t-u)} \frac{[\lambda(t-u)]^{j-\nu}}{(j-\nu)!}$$
$$dH_1(v-u) dH_1^{(j-1)}(v_1-v)$$

The first term is obtained by assuming that the server is performing a 2-task at t. The cycle of tasks in which the server is serving at t starts with i customers in unit 1 at t=0 and the unit 1 becomes empty after r services at time $u(0 < u \leq t)$. There are j arrivals in (u,t). The service of these j customers starts after the completion of the 2-task in progress. Let this 2-task end at time $v(t \leq v \leq t + x)$. The service completion in unit 1 of the j customers occur at time $v_1(v \leq v_1 \leq t + x)$. Now we integrate and sum over all choices of r, u, v, and v_1.

The second term is obtained by assuming that the server is performing a 1-task at t. The cycle of tasks in which the server is serving at t starts with i customers in unit 1 at t=o. Let there be r service completions in unit 1 before t. The last of these occurs at time u and at this time there are ν customers waiting in unit 1. There are j-ν arrivals in unit 1 in the interval (u,t) so that at t there are j-1 customers in unit 1 excepting the customer in service. The service completion, in unit 1, of the customer in service occurs at time v. The service completion, in unit 1, of the j-1 customers occurs at time v_1. Finally we sum over all choices of r, ν, u, v, and v_1.

Equation (11) is valid for all $j \geq 0$. For j=0 the last term disappears. Upon taking transform in (11) we find:

(12)

$$\psi_{ij}^{**}(\xi,s) = \sum_{r=i}^{\infty} g_{io}^{(r)}(\xi)h_1^j(s)\int_0^{\infty} e^{-sv}\int_0^v e^{-(\xi+\lambda-s)t}\,\frac{(\lambda t)^j}{j!}\,dt\,dH_2^{(r)}(v)$$

$$+ \sum_{r=0}^{\infty}\sum_{\nu=1}^{j} g_{i\nu}^{(r)}(\xi)h_1^{j-1}(s)\int_0^{\infty} e^{-sv}\int_0^v e^{-(\xi+\lambda-s)t}\,\frac{(\lambda t)^{j-\nu}}{(j-\nu)!}$$

$$dt\,dH_1(v)$$

Hence:

$$\psi_i(\xi,s,z) = \sum_{j=0}^{\infty} \psi_{ij}^{**}(\xi,s)\,z^j$$

$$= \frac{1}{\xi+\lambda-s-\lambda z h_1(s)}\Big\{\sum_{r=i}^{\infty} g_{io}^{(r)}(\xi)[h_2^r(s)-h_2^r(\xi+\lambda-\lambda z h_1(s))]$$

$$+ \sum_{r=0}^{\infty}\sum_{\nu=1}^{\infty} g_{i\nu}^{(r)}(\xi)h_1^{\nu-1}(s)z^{\nu}[h_1(s)-h_1(\xi+\lambda-\lambda z h_1(s))]\Big\}$$

Using (3) and (1.14) and simplifying we prove the above lemma.

Lemma 2.2

For $R(s) > 0$, $R(\xi) \geq 0$ and $|z| \leq 1$ the transform $\Phi_i(\xi,s,z)$ is given by:

(13)

$$\Phi_i(\xi,s,z) = \frac{1}{\xi+\lambda-s-\lambda z h_1(s)}\Big\{\sum_{n=1}^{\infty}\Big[{}_0q_i^{(n)}(\xi,\frac{\xi+\lambda-s}{\lambda}) - {}_0q_i^{(n)}(\xi,zh_1(s))\Big]$$

$$+ z\Big[h_1(s)-h_1(\xi+\lambda-\lambda z h_1(s))\Big]$$

$$\cdot \Big[zh_1(s) - h_1(\xi+\lambda-\lambda z h_1(s))\Big]^{-1}$$

$$\cdot \sum_{n=0}^{\infty}\Big[{}_0q_i^{(n)}(\xi,zh_1(s)) - {}_0q_i^{(n)}(\xi,\gamma_1(\xi))\Big]\Big\}$$

Proof:

As in lemma 1.7 we have:

$$(14)\quad \delta_i(\xi,s,z) = \sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} {}_oq_{i\nu}^{(n)}(\xi)\ \psi_\nu(\xi,s,z)$$

Substituting $\psi_i(\xi,s,z)$ from lemma 2.1 and simplifying the result follows. The convergence of the serices $\sum_{n=1}^{\infty} {}_oq_i^{(n)}(s,z)$ is discussed in Theorem 3 of Appendix B.

Theorem 2.1

For $R(s) > 0$ $R(\xi) \geq 0$ and $|z| \leq 1$, the transform $\theta_i(\xi,s,z)$ of the joint distribution $\theta_{ij}(t,x)$ of the queue length and virtual waiting time at t for the tandem queue with zero switching is given by:

(15)

$$\theta_i(\xi,s,z) = \delta_i(\xi,s,z) + \frac{\gamma^i(\xi)}{\xi+\lambda-\lambda\gamma(\xi)}\left[1+\lambda\delta_1(\xi,s,z)\right],\ i \geq 1,$$

where $\delta_i(\xi,s,z)$ is given by lemma 2.2.

Proof:

Similar to the proof of Theorem 1.1.

3. Distribution of Virtual Waiting time

The stochastic behavior of the process $\{\eta_1(t),\ 0 \leq t < \infty\}$ may be described as follows: $\eta_1(0)$ is the initial occupation time of the server. If $\eta_1(0+) = 0$ then the server is idle at $t=0+$. Let i be the initial queue length at $t=0$ and t_n the n-th arrival point and $X_{i+n}^{(\nu)}$ the service time, in

unit ν, of the customer arriving at time t_n. At t_n the value of $\eta_1(t)$ has a jump of magnitude $X_{i+n}^{(1)}$. Between any two arrivals $\eta_1(t)$ decreases linearly with slope -1. As soon as $\eta_1(t)$ reaches zero, it jumps suddenly to a magnitude equal to the total service time of all the customers present in unit 2 at that time, after which it linearly decreases with slop -1 until the arrival of a new customer. This is shown graphically in Figure 2.

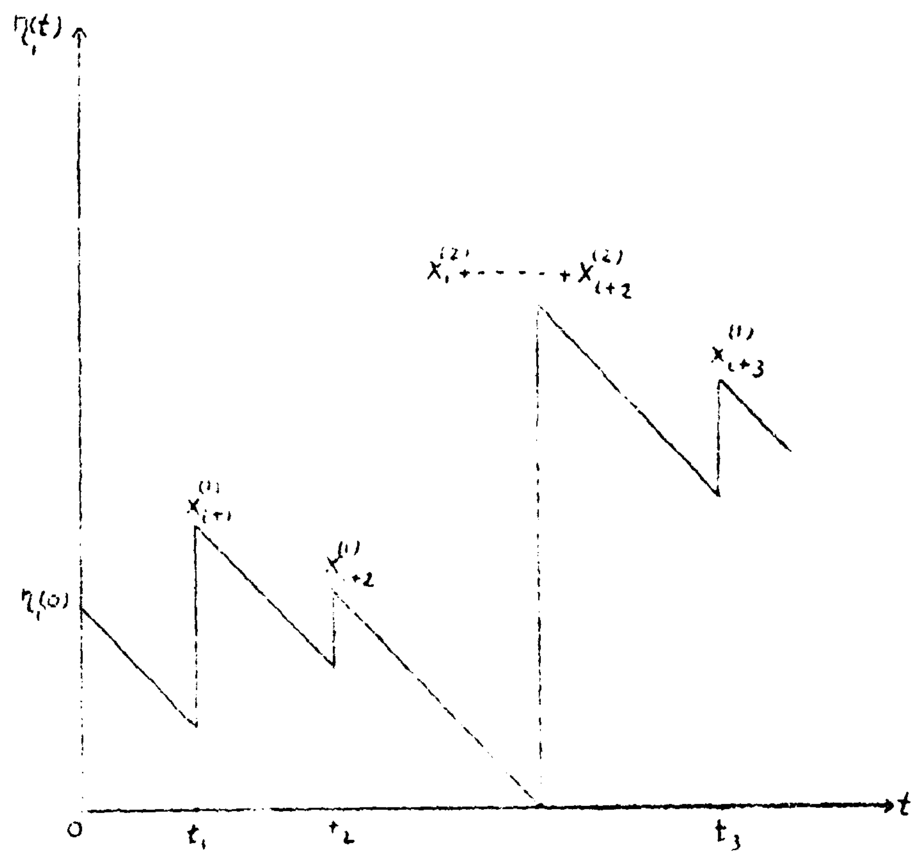

Figure 2

Graph of the Stochastic Behavior of the Process $\{\eta_1(t),\ 0 \leq t < \infty\}$

Let ${}_1W_i(t,x) = P\{\eta_1(t) \leq x \mid \xi(0) = i\}$ be the distribution function of the virtual waiting time $\eta_1(t)$, given that at t=0 there are $i \geq 1$ customers in unit 1. Let ${}_1W_i^{**}(\xi,s)$ be the transform of ${}_1W_i(t,x)$ defined in (1.53).

Theorem 2.2

For $R(s) > 0$ and $R(\xi) \geq 0$ the transform ${}_1W_i^{**}(\xi,s)$ of the distribution function ${}_1W_i(t,x)$ of the virtual waiting time $\eta_1(t)$ is given by:

$$(16) \qquad {}_1W_i^{**}(\xi,s) = {}_1\Lambda_i^{**}(\xi,s) + \frac{\gamma^i(\xi)}{\xi+\lambda-\lambda\gamma(\xi)}\left[1+\lambda \; {}_1\Lambda_1^{**}(\xi,s)\right]$$

where

$$(17) \qquad {}_1\Lambda_i^{**}(\xi,s) = \left[\xi+\lambda-s-\lambda h_1(s)\right]^{-1}\Big\{h_1^i(s)-\gamma^i(\xi)$$

$$+ \sum_{n=1}^{\infty}\left[a_n^i(\xi,\frac{\xi+\lambda-s}{\lambda})-a_n^i(\xi,\frac{\xi+\lambda}{\lambda})\right]\Big\} ,$$

and

$$a_o(\xi,z) = z$$

$$(18) \qquad a_n(\xi,z) = \gamma_1\{\xi,h_2[\xi+\lambda-\lambda a_{n-1}(\xi,z)]\} , \; n \geq 1,$$

Proof:

We get ${}_1W_i^{**}(\xi,s)$ by taking the limit $z \to 1$ in the transform $\theta_i(\xi,s,z)$ of the joint distribution $\theta_{ij}(t,x)$ of the queue length and virtual waiting time. Theorem 2.1 leads to:

$${}_1W_i^{**}(\xi,s) = \lim_{z \to 1} \theta_i(\xi,s,z)$$

$$(19) \qquad = \phi_i(\xi,s,1) + \frac{\gamma^i(\xi)}{\xi+\lambda-\lambda\gamma(\xi)}\left[1+\lambda \; \phi_1(\xi,s,1)\right]$$

Hence it suffices to prove that:

$$(20) \qquad \Phi_i(\xi,s,1) = {}_1\Lambda_i^{**}(\xi,s) \quad , \ i \geq 1 \ ,$$

From lemma 2.2 we have:

(21)

$$\Phi_i(\xi,s,1) = [\xi+\lambda-s-\lambda h_1(s)]^{-1} \left\{ h_1^i(s) + \sum_{n=1}^{\infty} {}_0q_i^{(n)}(\xi,\tfrac{\xi+\lambda-s}{\lambda}) - \sum_{n=0}^{\infty} {}_0q_i^{(n)}(\xi,\gamma_1(\xi)) \right\}$$

Equation (3) gives:

$$\begin{aligned} {}_0q_i^{(n)}(\xi,\gamma_1(\xi)) &= {}_0q_i^{(n)}(\xi,0) + \sum_{j=1}^{\infty} {}_0q_{ij}^{(n)}(\xi)\ \gamma_1^j(\xi) \\ &= {}_0q_i^{(n)}(\xi,0) + \sum_{j=1}^{\infty} {}_0q_{ij}^{(n)}(\xi) q_j(\xi,\tfrac{\xi+\lambda}{\lambda}) \\ &= {}_0q_i^{(n)}(\xi,0) + {}_0q_i^{(n+1)}(\xi,\tfrac{\xi+\lambda}{\lambda}) \end{aligned}$$

which upon summing over n leads to:

$$\begin{aligned} \sum_{n=0}^{\infty} {}_0q_i^{(n)}(\xi,\gamma_1(\xi)) &= \sum_{n=0}^{\infty} {}_0q_i^{(n)}(\xi,0) + \sum_{n=1}^{\infty} {}_0q_i^{(n)}(\xi,\tfrac{\xi+\lambda}{\lambda}) \\ &= \gamma^i(\xi) + \sum_{n=1}^{\infty} {}_0q_i^{(n)}(\xi,\tfrac{\xi+\lambda}{\lambda}) \end{aligned}$$

(For convergence of these series we refer to Theorem 3 in Appendix B)

Substitution of this in (21) yields:

(22)

$$\phi_1(\xi,s,1) = \left[\xi+\lambda-s-\lambda h_1(s)\right]^{-1}\left\{h_1^1(s)-\gamma^1(\xi) + \sum_{n=1}^{\infty}\left[{}_0q_1^{(n)}(\xi,\frac{\xi+\lambda-s}{\lambda}) - {}_0q_1^{(n)}(\xi,\frac{\xi+\lambda}{\lambda})\right]\right\}$$

Using lemma 2 of Appendix B and simplifying the above expression we prove (20).

Limiting Distribution of Virtual Waiting time

Let ${}_1W(x)$ be the limiting value of ${}_1W_1(t,x)$ as $t \to \infty$ and let ${}_1\omega(s)$ be its L.S.T. The existence of the limiting distribution can be proved by Zygmund's theorem as in Chap. I section 5.

Theorem 2.3

The L.S.T. of the limiting distribution ${}_1W(x)$ of the virtual waiting time $\eta_1(t)$ is given by:

(23)

$$ {}_1\omega(s) = \frac{(1-\lambda\alpha_1-\lambda\alpha_2)}{s-\lambda+\lambda h_1(s)}\left\{s + \lambda \sum_{n=1}^{\infty}\left[1-a_n(0,\frac{\lambda-s}{\lambda})\right]\right\} \quad \text{if } 1-\lambda\alpha_1-\lambda\alpha_2 > 0$$

$= 0$ otherwise

where the functions $a_n(\cdot,\cdot)$ are defined in (18).

Proof:

Similar to the proof of Theorem 1.3. As in (1.65) we obtain:

(24)

$$_1\omega(s) = (1-\lambda\alpha_1-\lambda\alpha_2)[1+\lambda\ {}_1\Lambda_1^{**}(0,s)] \quad \text{if } 1-\lambda\alpha_1-\lambda\alpha_2 > 0,$$

$$= 0 \quad \text{otherwise}$$

where $_1\Lambda_1^{**}(0,s)$ is given by (17):

$$_1\Lambda_1^{**}(0,s) = [\lambda-s-\lambda h_1(s)]^{-1}\left\{h_1(s)-1+\sum_{=1}^{\infty}[a_n(0,\frac{\lambda-s}{\lambda}) - a_n(0,1)]\right\}$$

$$(25) \qquad = \frac{1}{s-\lambda+\lambda h_1(s)}\left\{1-h_1(s)+\sum_{n=1}^{\infty}[1-a_n(0,\frac{\lambda-s}{\lambda})]\right\}$$

The convergence of the series $\sum_{n=1}^{\infty}[1-a_n(0,\frac{\lambda-s}{\lambda})]$ is discussed in lemma 7 of Appendix B.

Substituion of (25) in (24) proves the theorem.

Taking the limit as $s \to 0$ in (23) we observe that:

$$(26) \qquad _1\omega(0) = \frac{(1-\lambda\alpha_1-\lambda\alpha_2)}{(1-\lambda\alpha_1)}\left[1+\sum_{n=1}^{\infty}a_n'(0,1)\right]$$

$$= 1 \qquad \text{if } 1-\lambda\alpha_1-\lambda\alpha_2 > 0,$$

by lemma 1 of Appendix B.

Expected Value of the Limiting Distribution

of Virtual Waiting time

Let M_{η_1} denote the expected value of the limiting distribution of $\eta_1(t)$. Taking the derivative of (23) results in:

$$M_{\eta_1} = -\frac{\partial_1 \omega(s)}{\partial s}\Big]_{s=0}$$

$$(27) \qquad = \frac{(1-\lambda\alpha_1-\lambda\alpha_2)}{2(1-\lambda\alpha_1)^2}\left\{\lambda\,\beta_1[1+\sum_{n=1}^{\infty} a_n'(0,1)] - \frac{(1-\lambda\alpha_1)}{\lambda}\sum_{n=1}^{\infty} a_n''(0,1)\right\}$$

Using lemma 1 of Appendix B and simplifying (27) we obtain:

$$(28) \qquad M_{\eta_1} = \frac{\lambda[\beta_1+\beta_2+2(\frac{\lambda\alpha_2}{1-\lambda\alpha_1})\alpha_1\alpha_2]}{2(1-\lambda\alpha_1)[1-(\frac{\lambda\alpha_2}{1-\lambda\alpha_1})^2]}$$

Computation of higher moments seems to be very tedious.

If we denote $M_\eta = \frac{\lambda(\beta_1+\beta_2+2\alpha_1\alpha_2)}{2(1-\lambda\alpha_1-\lambda\alpha_2)}$ which is the steady state expected ~~virtual~~ waiting time of an $M|G|1$ queue with service time distribution $H_1 * H_2(\cdot)$, then it can be shown that:

$$\frac{1}{2}M_\eta - \frac{\alpha_1}{2} < M_{\eta_1} < c_1 M_\eta - c_2\,\alpha_1, \quad \text{for any } c_1 > \frac{1}{2},\ c_2 < \frac{1}{2},$$

4. Distribution of Queue length

Let $P_{ij}(t)=P\{\xi(t)=j \mid \xi(0)=i\}$ and $\pi_{ij}(\xi)$ be its Laplace transform and

$$(29) \qquad \pi_i(\xi,z) = \sum_{j=0}^{\infty} \pi_{ij}(\xi)z^j, \quad R(\xi)>0,\ |z|\le 1 \quad \text{or } R(\xi)\ge 0,\ |z|<1$$

Theorem 2.4

The generating function $\pi_i(\xi,z)$ is given by:

(30)

$$\pi_i(\xi,z) = X_i(\xi,z) + \frac{Y^i(\xi)}{\xi+\lambda-\lambda Y(\xi)} [1 + \lambda X_i(\xi,z)] , \quad i \geq 1,$$

where $X_i(\xi,z)$ is given by:

(31)

$$X_i(\xi,z) = \frac{1}{\xi+\lambda-\lambda z} \Big\{ \sum_{n=1}^{\infty} \left[a_n^i(\xi,\tfrac{\xi+\lambda}{\lambda}) - a_n^i(\xi,z)\right]$$

$$+ z\left[z-h_1(\xi+\lambda-\lambda z)\right]^{-1}\left[1-h_1(\xi+\lambda-\lambda z)\right]$$

$$\sum_{n=0}^{\infty} \left[a_n^i(\xi,z) - a_n^i(\xi,Y_1(\xi))\right] \Big\} \quad , i \geq 1 ,$$

and the functions $a_n(\cdot,\cdot)$ are defined in (18).

<u>Proof</u>:

Analogous to equation (1.84) we obtain:

(32) $$\pi_i(\xi,z) = \Phi_i(\xi,0,z) + \frac{Y^i(\xi)}{\xi+\lambda-\lambda Y(\xi)}\left[1+\lambda\ \Phi_i(\xi,0,z)\right]$$

where $\Phi_i(\xi,0,z)$ is obtained from lemma 2.2:

$$\Phi_i(\xi,0,z) = \frac{1}{\xi+\lambda-\lambda z} \Big\{ \sum_{n=1}^{\infty}\left[{}_oq_i^{(n)}(\xi,\tfrac{\xi+\lambda}{\lambda}) - {}_oq_i^{(n)}(\xi,z)\right]$$

$$+ z\left[z-h_1(\xi+\lambda-\lambda z)\right]^{-1}\left[1-h_1(\xi+\lambda-\lambda z)\right]$$

$$\sum_{n=0}^{\infty}\left[{}_oq_i^{(n)}(\xi,z) - {}_oq_i^{(n)}(\xi,Y_1(\xi))\right] \Big\}$$

$$= X_i(\xi,z)$$

(by lemma 2 in Appendix B)

which together with (32) proves (30).

Limiting Distribution of Queue length

Let $P_j^* = \lim_{t \to \infty} P_{ij}(t)$

and

(33) $$P^*(z) = \sum_{j=0}^{\infty} P_j^* z^j \quad , |z| < 1,$$

where the existence of the limit is established as in Chap. I section 6.

Theorem 2.5

The generating function $P^*(z)$ is given by:

(34)

$$P^*(z) = \frac{(1-\lambda\alpha_1-\lambda\alpha_2)h_1(\lambda-\lambda z)}{h_1(\lambda-\lambda z)-z} \sum_{n=0}^{\infty} \left[1-a_n(0,z)\right] \text{ if } 1-\lambda\alpha_1-\lambda\alpha_2 > 0 ,$$

$$= 0 \text{ otherwise,}$$

where the functions $a_n(\cdot,\cdot)$ are defined in (18).

Proof:

It follows as in (1.92) that:

(35) $$P^*(z) = \lim_{\xi \to 0} \xi \pi_1(\xi,z)$$

$$= (1-\lambda\alpha_1-\lambda\alpha_2) \left[1 + \lambda X_1(0,z)\right] \text{ if } 1-\lambda\alpha_1-\lambda\alpha_2 > 0,$$

$$= 0 \text{ otherwise,}$$

where $X_1(0,z)$ is given by (31):

$$(36)\ X_1(o,z) = \frac{1}{\lambda-\lambda z}\left\{\sum_{n=1}^{\infty}\left[a_n(0,1) - a_n(0,z)\right]\right.$$

$$+ z\left[z-h_1(\lambda-\lambda z)\right]^{-1}\left[1-h_1(\lambda-\lambda z)\right]$$

$$\left.\sum_{n=0}^{\infty}\left[a_n(0,z) - a_n(0,1)\right]\right\}$$

$$= \frac{-1}{\lambda}\left\{1 + \frac{h_1(\lambda-\lambda z)}{z-h_1(\lambda-\lambda z)}\sum_{n=0}^{\infty}\left[1-a_n(0,z)\right]\right\}$$

Substitution of (36) in (35) proves the theorem.

It can be shown that:

$$P^*(1) = 1$$

Expected Value of the Limiting Distribution of Queue length

Let M_ξ denote the expected value of the limiting distribution of $\xi(t)$. From Theorem 2.5 we obtain:

$$M_\xi = \frac{\partial}{\partial z} P^*(z)\Big]_{z=1}$$

$$= \frac{(1-\lambda\alpha_1-\lambda\alpha_2)}{2(1-\lambda\alpha_1)^2}\left\{\left[2\lambda\alpha_1(1-\lambda\alpha_1)+\lambda^2\beta_1\right]\sum_{n=0}^{\infty} a_n'(0,1)\right.$$

$$\left.+ (1-\lambda\alpha_1)\sum_{n=0}^{\infty} a_n''(0,1)\right\}$$

$$(37)\qquad = \frac{\lambda^2(\beta_1+\beta_2)+2\lambda\alpha_1(1-\lambda\alpha_1)}{2(1-\lambda\alpha_1)\left[1-\left(\frac{\lambda\alpha_2}{1-\lambda\alpha_1}\right)^2\right]}$$

(by lemma 1 in Appendix B)

$$= \lambda\,\alpha_1 + \lambda\,M_{\eta_1}$$

5. Distribution of Total Time in the System

Consider all the customers present in unit 1 at time t. Their total time required to complete services in unit 1 as well as in unit 2 is defined as the total time in the system at t. We denote by $\eta_2(t)$ the total time in the system at t for the zero switching case.

Let ${}_2W_i(t,x) = P\{\eta_2(t) \le x \mid \xi(0) = i\}$ be the distribution function of the total time in the system $\eta_2(t)$, given that the queue length at t=0 is i.

Further we denote:

(38)

$$ {}_2\Lambda_i(t,x)=P\left\{0 < \eta_2(t) \le x, \eta_2(\tau)\ne 0 \text{ for all } \tau\in(0,t] \mid \xi(0) = i\right\} $$

and let ${}_2X_j(t,x)$ be the probability that at time t the original cycle has not ended and that $0 < \eta_2(t) \le x$ and $\eta_2(\tau) \ne 0$ for all $\tau\in(0,t]$, given that at t=0 the service started in unit 1 with i customers. Analogous to equations (1.38) and (1.39) we obtain:

(39)

$$ {}_2W_i(t,x) = {}_2\Lambda_i(t,x) + \int_0^t {}_2\Lambda_1(t-u,x)\, dM_i(u) $$

$$ + P\left\{\eta_2(t) = 0 \mid \xi(0) = i\right\} U(x) $$

and

(40)

$$ {}_2\Lambda_i(t,x) = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \int_0^t d_0Q_{ij}^{(n)}(u)\ {}_2X_j(t-u,x) $$

For $R(s) > 0$ and $R(\xi) \geq 0$ we define the following transforms:

$$ {}_2W_j^{**}(\xi,s) = \int_0^\infty e^{-\xi t} \int_0^\infty e^{-sx} \, d_2W_j(t,x) \, dt $$

$$ {}_2\Lambda_j^{**}(\xi,s) = \int_0^\infty e^{-\xi t} \int_0^\infty e^{-sx} \, d_2\Lambda_j(t,x) \, dt $$

$$ {}_2X_j^{**}(\xi,s) = \int_0^\infty e^{-\xi t} \int_0^\infty e^{-sx} \, d_2X_j(t,x) \, dt $$

Lemma 2.3

For $R(s) > 0$ and $R(\xi) \geq 0$ the transform ${}_2X_j^{**}(\xi,s)$ is given by:

(41)

$$ {}_2X_j^{**}(\xi,s) = \left[\xi-s+\lambda\gamma_1(s)-\lambda h_2(s)\gamma_1(s)\right]^{-1}\Big\{q_j\Big[\xi,\frac{\xi-s+\lambda\gamma_1(s)}{\lambda}\Big] $$

$$ - q_j[\xi,h_2(s)\gamma_1(s)] + $$

$$ \left[\gamma_1(s)-h_1(\xi+\lambda-\lambda h_2(s)\gamma_1(s))\right]^{-1} $$

$$ \left[h_1(s+\lambda-\lambda\gamma_1(s))-h_1(\xi+\lambda-\lambda h_2(s)\gamma_1(s))\right] $$

$$ \left[\left(h_2(s)\gamma_1(s)\right)^j - q_j\left(\xi,\frac{\xi+\lambda-s}{\lambda}\right)\right]\Big\} , \; j \geq 1 . $$

Proof:

The probability ${}_2X_j(t,x)$ is given in terms of the probabilities $G_{ij}^{(n)}(\cdot)$ by:

(42)

$$
{}_2X_j(t,x)=\sum_{r=j}^{\infty}\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}\sum_{r_1=\nu_1+\nu_2}^{\infty}\int_{\substack{o\\(u)}}^{t}\int_{\substack{t\\(v)}}^{t+x}\int_{\substack{v\\(v_1)}}^{t+x}\int_{\substack{v_1\\(v_2^1)}}^{t+x} d\,G_{jo}^{(r)}(u)
$$

$$
\cdot\ e^{-\lambda(t-u)}\frac{[\lambda(t-u)]^{\nu_1}}{\nu_1!}\,e^{-\lambda(v-t)}\frac{[\lambda(v-t)]^{\nu_2}}{\nu_2!}\,d_vH_2^{(r)}(v-u)
$$

$$
\cdot\ d_{v_1}G_{\nu_1+\nu_2,o}^{(r_1)}(v_1-v)\ d_{v_2}H_2^{(\nu_1)}(v_2-v_1)
$$

$$
+\sum_{r=0}^{\infty}\sum_{\nu=1}^{\infty}\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}\sum_{r_1=\nu+\nu_1+\nu_2-1}^{\infty}\int_{\substack{o\\(u)}}^{t}\int_{\substack{t\\(v)}}^{t+x}\int_{\substack{v\\(v_1)}}^{t+x}\int_{\substack{v_1\\(v_2^1)}}^{t+x} dG_{j\nu}^{(r)}(u)
$$

$$
\cdot\ e^{-\lambda(t-u)}\frac{[\lambda(t-u)]^{\nu_1}}{\nu_1!}\,e^{-\lambda(v-t)}\frac{[\lambda(v-t)]^{\nu_2}}{\nu_2!}\,d_vH_1(v-u)
$$

$$
\cdot\ d_{v_1}G_{\nu+\nu_1+\nu_2-1,o}^{(r_1)}(v_1-v)\ d_{v_2}H_2^{(r+\nu+\nu_1)}(v_2-v_1)
$$

<u>The first term</u> is obtained by assuming that the server is performing a 2-task at time t. The cycle of tasks in which the server is serving at t starts with j customers in unit 1 at t=0, and the unit 1 becomes empty after r services at time u. The number of arrivals in unit 1 between times u and t is ν_1. The r customers in unit 2 at time u have service completion at time v and ν_2 is the number of arrivals in unit 1 between times t and v. At time v there are $\nu_1+\nu_2$ customers in unit 1.

Starting with $\nu_1+\nu_2$ customers at time v, unit 1 becomes empty after r_1 services at time v_1. The service completion of the ν_1 customers in unit 2 occurs at time v_2. Finally we integrate and sum over all choices of r, ν_1, ν_2, r_1, u, v, v_1, and v_2.

The second term is obtained by assuming that the server is performing a 1-task at time t. The cycle of tasks in which the server is serving at t starts with j customers in unit 1 at t=0. There are r service completions in unit 1 before t. The last service completion before t occurs at time u, at which there are ν customers waiting in unit 1. The number of arrivals in unit 1 between times u and t is ν_1. The service completion of the customer in service at t occurs at time v and there are ν_2 arrivals between t and v. At time v there are $\nu+\nu_1+\nu_2-1$ customers waiting in unit 1. Starting with these $\nu+\nu_1+\nu_2-1$ customers at time v, unit 1 becomes empty after r_1 services at time v_1. Lastly the service completion of the $r+\nu+\nu_1$ customers, who arrived in unit 1 up to time t, occurs in unit 2 at time v_2. Now we sum and integrate over all choices of r, ν, ν_1, ν_2, r_1, u, v, v_1, and v_2.

Taking the transform of (42):

(43)

$$
{}_2X_j^{**}(\xi,s) = \sum_{r=j}^{\infty}\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}\sum_{r_1=\nu_1+\nu_2}^{\infty} g_{jo}^{(r)}(\xi)h_2^{\nu_1}(s)\, g_{\nu_1+\nu_2,o}^{(r_1)}(s)
$$

$$
\cdot \int_0^{\infty} e^{-(\xi+\lambda)t}\,\frac{(\lambda t)^{\nu_1}}{\nu_1!}\,dt \int_0^{\infty} e^{-(s+\lambda)v}\frac{(\lambda v)^{\nu_2}}{\nu_2!}\,d_vH_2^{(r)}(v+t)
$$

$$
+ \sum_{r=0}^{\infty}\sum_{\nu=1}^{\infty}\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}\sum_{r_1=\nu+\nu_1+\nu_2-1}^{\infty} g_{j\nu}^{(r)}(\xi)h_2^{r+\nu+\nu_1}(s)g_{\nu+\nu_1+\nu_2-1,o}^{(r_1)}(s)
$$

$$
\cdot \int_0^{\infty} e^{-(\xi+\lambda)t}\,\frac{(\lambda t)^{\nu_1}}{\nu_1!}\,dt \int_0^{\infty} e^{-(s+\lambda)v}\,\frac{(\lambda v)^{\nu_2}}{\nu_2!}\,d_vH_1(v+t)
$$

Using lemma 1.1 and simplifying the above expression leads to:

(44)

$$
{}_2X_j^{**}(\xi,s) = \sum_{r=j}^{\infty} g_{jo}^{(r)}(\xi)\int_0^{\infty} e^{-[\xi+\lambda-\lambda h_2(s)\gamma_1(s)]t}\,dt
$$

$$
\cdot \int_0^{\infty} e^{-[s+\lambda-\lambda\gamma_1(s)]v}\,d_vH_2^{(r)}(v+t)
$$

$$
+ \sum_{r=0}^{\infty}\sum_{\nu=1}^{\infty} g_{j\nu}^{(r)}(\xi)h_2^{r+\nu}(s)\gamma_1^{\nu-1}(s)\int_0^{\infty} e^{-[\xi+\lambda-\lambda h_2(s)\gamma_1(s)]t}\,dt
$$

$$
\cdot \int_0^{\infty} e^{-[s+\lambda-\lambda\gamma_1(s)]v}\,d_vH_1(v+t)
$$

$$
= \sum_{r=j}^{\infty} g_{jo}^{(r)}(\xi)\int_0^{\infty} e^{-[s+\lambda-\lambda\gamma_1(s)]v}
$$

$$
\cdot \int_0^{v} e^{-[\xi-s+\lambda\gamma_1(s)-\lambda h_2(s)\gamma_1(s)]t}\,dt\,dH_2^{(r)}(v)
$$

$$+ \sum_{r=0}^{\infty} \sum_{\nu=1}^{\infty} g_{j\nu}^{(r)}(\xi) h_2^{r+\nu}(s) \gamma_1^{\nu-1}(s) \int_0^{\infty} e^{-[s+\lambda-\lambda\gamma_1(s)]v}$$

$$\cdot \int_0^v e^{-[\xi-s+\lambda\gamma_1(s)-\lambda h_2(s)\gamma_1(s)]t} dt \; dH_2^{(r)}(v)$$

$$= \Big[\xi-s+\lambda\gamma_1(s)-\lambda h_2(s)\gamma_1(s)\Big]^{-1} \Big\{ \sum_{r=j}^{\infty} g_{jo}^{(r)}(\xi) \Big[h_2^r(s+\lambda-\lambda\gamma_1(s))$$

$$- h_2^r(\xi+\lambda-\lambda h_2(s)\gamma_1(s)) \Big]$$

$$+ \sum_{r=0}^{\infty} \sum_{\nu=1}^{\infty} g_{j\nu}^{(r)}(\xi) h_2^{r+\nu}(s) \gamma_1^{\nu-1}(s) \Big[h_1(s+\lambda-\lambda\gamma_1(s))$$

$$- h_1(\xi+\lambda-\lambda h_2(s)\gamma_1(s)) \Big] \Big\}$$

Using (3) and (1.14) and simplifying further:

(45)

$$ {}_2\chi_j^{**}(\xi,s) = \Big[\xi-s+\lambda\gamma_1(s)-\lambda h_2(s)\gamma_1(s)\Big]^{-1} \Big\{ q_j[\xi, \frac{\xi-s+\lambda\gamma_1(s)}{\lambda}]$$

$$- q_j[\xi, h_2(s)\gamma_1(s)] + \Big[\gamma_1(s) - h_1(\xi+\lambda-\lambda h_2(s)\gamma_1(s))\Big]^{-1}$$

$$\cdot \Big[\big(h_2(s)\gamma_1(s)\big)^j - \gamma_1^j(\xi, h_2(s)) \Big] \Big[h_1(s+\lambda-\lambda\gamma_1(s))$$

$$- h_1(\xi+\lambda-\lambda h_2(s)\gamma_1(s)) \Big] \Big\}$$

This together with $\gamma_1^j(\xi, h_2(s)) = q_j(\xi, \frac{\xi+\lambda-s}{\lambda})$ proves the lemma.

Lemma 2.4

For $R(s) > 0$ and $R(\xi) \geq 0$ the transform ${}_2\Lambda_i^{**}(\xi,s)$ is given by:

(46)

$$
\begin{aligned}
{}_2\Lambda_i^{**}(\xi,s) = \Big[\xi-s+\lambda\gamma_1(s)-\lambda h_2(s)\gamma_1(s)\Big]^{-1}\Big\{\sum_{n=1}^{\infty}\Big[a_n^i\Big(\xi,\frac{\xi-s+\lambda\gamma_1(s)}{\lambda}\Big) \\
- a_n^i(\xi,h_2(s)\gamma_1(s))\Big] + \Big[\gamma_1(s)-h_1(\xi+\lambda-\lambda h_2(s)\gamma_1(s))\Big]^{-1} \\
\cdot \Big[h_1(s+\lambda-\lambda\gamma_1(s))-h_1(\xi+\lambda-\lambda h_2(s)\gamma_1(s))\Big] \\
\cdot \Big[(h_2(s)\gamma_1(s))^i-\gamma^i(\xi) + \sum_{n=1}^{\infty}\Big(a_n^i(\xi,h_2(s)\gamma_1(s)) \\
- a_n^i\Big(\xi,\frac{\xi+\lambda-s}{\lambda}\Big)\Big)\Big]\Big\}
\end{aligned}
$$

where the functions $a_n(\cdot,\cdot)$ are defined in (18).

Proof:

Upon transformation of (40) yields:

$$
{}_2\Lambda_i^{**}(\xi,s) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} {}_0q_{ij}^{(n)}(\xi)\ {}_2X_j^{**}(\xi,s)
$$

Substitution of lemma 2.3 results in:

(47)

$$
{}_2\Lambda_i^{**}(\xi,s) = \left[\xi-s+\lambda\gamma_1(s)-\lambda h_2(s)\gamma_1(s)\right]^{-1}\Big\{\sum_{n=1}^{\infty}\Big[{}_oq_i^{(n)}\big(\xi,\frac{\xi-s+\lambda\gamma_1(s)}{\lambda}\big)
$$

$$
- {}_oq_i^{(n)}(\xi,h_2(s)\gamma_1(s))\Big]+\left[\gamma_1(s)-h_1(\xi+\lambda-\lambda h_2(s)\gamma_1(s))\right]^{-1}
$$

$$
\cdot\left[h_1(s+\lambda-\lambda\gamma_1(s))-h_1(\xi+\lambda-\lambda h_2(s)\gamma_1(s))\right]
$$

$$
\cdot\Big[(h_2(s)\gamma_1(s))^i-\sum_{n=1}^{\infty}{}_oq_i^{(n)}(\xi,0)+\sum_{n=1}^{\infty}\Big({}_oq_i^{(n)}(\xi,h_2(s)\gamma_1(s))
$$

$$
- {}_oq_i^{(n)}\big(\xi,\frac{\xi+\lambda-s}{\lambda}\big)\Big)\Big]\Big\}
$$

This and lemma 2 in Appendix B together with $\sum_{n=1}^{\infty} {}_oq_i^{(n)}(\xi,0) = \gamma^i(\xi)$ prove the lemma.

<u>Theorem 2.5</u>

For $R(s) > 0$ and $R(\xi) \geq 0$ the transform ${}_2W_i^{**}(\xi,s)$ of the distribution function ${}_2W_i(t,x)$ of the total time in the system is given by:

(48)

$$
{}_2W_i^{**}(\xi,s) = {}_2\Lambda_i^{**}(\xi,s) + \frac{\gamma^i(\xi)}{\xi+\lambda-\lambda\gamma(\xi)}\left[1 + \lambda\ {}_2\Lambda_1^{**}(\xi,s)\right],\ i \geq 1,
$$

where ${}_2\Lambda_i^{**}(\xi,s)$ is given by lemma 2.4.

<u>Proof:</u>

Similar to the proof of Theorem 2.2.

Limiting Distribution of the Total Time in the System

Let ${}_2W(x) = \lim_{t \to \infty} {}_2W_1(t,x)$ and ${}_2\omega(s)$ be its L.S.T.

The existence of the limiting distribution can be proved as in Chap. I section 5.

Theorem 2.6

The L.S.T. of the limiting distribution of the total time in the system is given by:

$$(49) \qquad {}_2\omega(s) = (1-\lambda\alpha_1-\lambda\alpha_2)[1+\lambda\ {}_2\Lambda_1^{**}(0,s)] \text{ if } 1-\lambda\alpha_1-\lambda\alpha_2 > 0,$$

$$= 0 \quad \text{otherwise,}$$

where ${}_2\Lambda_1^{**}(0,s)$ is given by:

(50)

$$\begin{aligned}
{}_2\Lambda_1^{**}(0,s) = {} & \left[-s+\lambda\gamma_1(s)-\lambda h_2(s)\gamma_1(s)\right]^{-1} \Big\{ \sum_{n=1}^{\infty} \Big[a_n\Big(0,\frac{\lambda\gamma_1(s)-s}{\lambda}\Big) \\
& - a_n(0,h_2(s)\gamma_1(s)) \Big] + \\
& \left[\gamma_1(s)-h_1(\lambda-\lambda h_2(s)\gamma_1(s))\right]^{-1} \\
& \cdot \left[h_1(s+\lambda-\lambda\gamma_1(s))-h_1(\lambda-\lambda h_2(s)\gamma_1(s))\right] \\
& \cdot \Big[h_2(s)\gamma_1(s)-1 + \sum_{n=1}^{\infty}\Big(a_n(0,h_2(s)\gamma_1(s))-a_n\Big(0,\frac{\lambda-s}{\lambda}\Big)\Big)\Big] \Big\}
\end{aligned}$$

and the functions $a_n(\cdot,\cdot)$ are defined in (18).

Proof:

Equation (49) is obtained as in Theorem 2.3 and (50) from lemma 2.4 by taking $\xi \to 0$.

In (50) taking the limit $s \to 0$, we observe that:

$$(51) \qquad {}_2\Lambda_1^{**}(0,0) = \frac{\alpha_2 + \frac{\alpha_1}{1-\lambda\alpha_1} \sum_{n=0}^{\infty} a_n'(0,1)}{(1 - \lambda\, \alpha_2)}$$

$$= \frac{\alpha_1 + \alpha_2}{1-\lambda\alpha_1-\lambda\alpha_2}$$

(by lemma 1 in Appendix B)

Now from (49) it is easily seen that:

$${}_2\omega(0) = 1 \qquad \text{if } 1-\lambda\alpha_1-\lambda\alpha_2 > 0.$$

The Steady State Expected Value of the Total

Time in the System

Let M_{η_2} denote the expected value of the limiting distribution of $\eta_2(t)$. From (49) we have:

$$M_{\eta_2} = - \frac{\partial}{\partial s}\, {}_2\omega(s)\Big]_{s=0}$$

$$(52) \qquad = - \lambda(1-\lambda\alpha_1-\lambda\alpha_2)\, \frac{\partial}{\partial s}\, {}_2\Lambda_1^{**}(0,s)\Big]_{s=0}$$

To find the derivative $\frac{\partial}{\partial s}\, {}_2\Lambda_1^{**}(0,s)\Big]_{s=0}$ we will proceed as follows:

Let $\Delta_1(s)$ and $\Delta_2(s)$ denote respectively the numerator and denominator of ${}_2\Lambda_1^{**}(0,s)$. That is:

(53)

$$\Delta_1(s) = \left[\gamma_1(s)-h_1(\lambda-\lambda h_2(s)\gamma_1(s))\right] \sum_{n=1}^{\infty}\left[a_n(0,\frac{\lambda\gamma_1(s)-s}{\lambda})\right.$$

$$\left. - a_n(0,h_2(s)\gamma_1(s))\right] + \left[h_1(s+\lambda-\lambda\gamma_1(s))\right.$$

$$\left. - h_1(\lambda-\lambda h_2(s)\gamma_1(s))\right]\left[h_2(s)\gamma_1(s)-1\right.$$

$$\left. + \sum_{n=1}^{\infty}\left(a_n(0,h_2(s)\gamma_1(s))-a_n(0,\frac{\lambda-s}{\lambda})\right)\right]\}$$

(54)

$$\Delta_2(s) = \left[-s+\lambda\gamma_1(s)-\lambda h_2(s)\gamma_1(s)\right]\left[\gamma_1(s)-h_1(\lambda-\lambda h_2(s)\gamma_1(s))\right]$$

(55) $$\frac{\partial}{\partial s}\, {}_2\Lambda_1^{**}(0,s)\Big]_{s=0} = \frac{\Delta_2(s)\ \Delta_1'(s) - \Delta_1(s)\ \Delta_2'(s)}{[\Delta_2(s)]^2}\Big]_{s=0}$$

Applying l'Hopital's rule four times:

(56) $$\frac{\partial}{\partial s}\, {}_2\Lambda_1^{**}(0,s)\Big]_{s=0} = \frac{\Delta_2''(0)\ \Delta_1'''(0) - \Delta_1''(0)\ \Delta_2'''(0)}{3[\Delta_2''(0)]^2}$$

where the number of primes indicates the number of successive derivatives taken with respect to s.

Computations yield:

(57) $$\Delta_1''(0) = \frac{2\,\alpha_1(\alpha_1 + \alpha_2)(1 - \lambda\alpha_2)^2}{1 - \lambda\,\alpha_1 - \lambda\,\alpha_2}$$

$$\Delta_1'''(0) = \frac{-3(1-\lambda\alpha_2)(\alpha_1+\alpha_2)}{1-\lambda\alpha_1-\lambda\alpha_2}\Big\{\frac{\beta_1}{(1-\lambda\alpha_1)^2} - \beta_1\,\lambda^2(\alpha_2 + \frac{\alpha_1}{1-\lambda\alpha_1})^2$$

$$- \lambda\alpha_1\,(\beta_2 + \frac{2\alpha_1\alpha_2}{1-\lambda\alpha_1})\Big\}$$

$$- 3\alpha_1(1-\lambda\alpha_2)\Big\{\frac{\alpha_1(2-\lambda\alpha_1)}{\lambda(1-\lambda\alpha_1)^2}\sum_{n=1}^{\infty} a_n''(0,1) + \beta_2$$

$$+ \frac{2\alpha_1\alpha_2}{1-\lambda\alpha_1} + \frac{\beta_1}{(1-\lambda\alpha_1)^2(1-\lambda\alpha_1-\lambda\alpha_2)}\Big\}$$

(58) $$\Delta_2''(0) = 2\alpha_1(1-\lambda\alpha_2)^2$$

$$\Delta_2'''(0) = -\,3(1-\lambda\alpha_2)\Big\{\frac{\beta_1}{(1-\lambda\alpha_1)^2} - \beta_1\lambda^2(\alpha_2 + \frac{\alpha_1}{1-\lambda\alpha_1})^2$$

$$- 2\lambda\alpha_1(\beta_2 + \frac{2\alpha_1\,\alpha_2}{1-\lambda\alpha_1})\Big\}$$

Substituting these values in (56) and simplifying we obtain:

(59)

$$\frac{\partial}{\partial s}\,{}_2\Lambda_1^{**}(0,s)\Big]_{s=0} = \frac{-1}{2(1-\lambda\alpha_2)}\Big\{\frac{\alpha_1(2-\lambda\alpha_1)}{\lambda(1-\lambda\alpha_1)^2}\sum_{n=1}^{\infty} a_n''(0,1) +$$

$$+ \frac{\beta_1 + \beta_2(1-\lambda\alpha_1)^2 + 2\alpha_1\alpha_2(1-\lambda\alpha_1)}{(1-\lambda\alpha_1)^2(1-\lambda\alpha_1-\lambda\alpha_2)} \Big\}$$

$$= - \left\{2(1-\lambda\alpha_1)^3[1-(\frac{\lambda\alpha_2}{1-\lambda\alpha_1})^2]\ (1-\lambda\alpha_1-\lambda\alpha_2)\right\}^{-1}$$

$$\left\{(1-\lambda\alpha_1)(1+\lambda\alpha_2)(\beta_1+\beta_2)+2\alpha_1\alpha_2[(1-\lambda\alpha_1)^2+\lambda\ \alpha_2]\right\} .$$

(by lemma 1 in Appendix B)

Equation (52) together with (59) yields:

(60)

$$M_{\eta_2} = \frac{\lambda\{(1-\lambda\alpha_1)(1+\lambda\alpha_2)(\beta_1+\beta_2)+2\alpha_1\alpha_2[(1-\lambda\alpha_1)^2+\lambda\alpha_2]\}}{2(1 - \lambda\alpha_1)^3[1 - (\frac{\lambda\alpha_2}{1-\lambda\alpha_1})^2]}$$

6. The State of the Server

In this section we try to answer questions of the type: (i) What is the probability that at time t the server is busy (or idle)? (ii) If the server is busy at t what is the probability that he is serving in unit 1 (or unit 2)? (iii) And if he is serving in unit 1 (or unit 2) at t what is the probability that he is serving the r-th customer of the cycle in unit 1 (or unit 2)?

Finally we study the limiting behavior of the above probabilities.

We define the following probabilities:

$\theta_{\nu r}(t/i)$ is the probability that at t the r-th customer of a cycle of tasks in unit ν, $\nu=1,2$, is being served, given that the service started at t=0 with $i > 0$ customers in unit 1, and its transform:

(61) $$\theta^*_{\nu r}(\xi/i) = \int_0^\infty e^{-\xi t}\, \theta_{\nu r}(t/i)\, dt, \qquad \nu=1,2,$$

Further, $\theta_\nu(t/i)$ is the probability that at t the server is serving in unit ν, $\nu=1,2$, given that at t=0 there were $i > 0$ customers in unit 1. Then:

(62) $$\theta_\nu(t/i) = \sum_{r=1}^{\infty} \theta_{\nu r}(t/i)\ ; \qquad \nu=1,2,$$

Lemma 2.5

The transforms $\theta^*_{\nu r}(\xi/i)$ of $\theta_{\nu r}(t/i)$ are given by:

(63)
$$\theta^*_{1r}(\xi|i) = \frac{1}{\xi}\left[1-h_1(\xi)\right] \sum_{j=1}^{\infty} {}_o r_{ij}(\xi) \left[g_j^{(r-1)}(\xi,1)-g_j^{(r-1)}(\xi,0)\right]$$

$$+ \frac{m_1(\xi)}{\xi}\left[1-h_1(\xi)\right] \sum_{j=1}^{\infty} {}_o r_{1j}(\xi) \left[g_j^{(r-1)}(\xi,1)-g_j^{(r-1)}(\xi,0)\right]$$

(64)
$$\theta^*_{2r}(\xi|i) = \frac{1}{\xi}\left[1-h_2(\xi)\right] \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} I_{[r\leq n]}\, {}_o r_{ij}(\xi) g_j^{(n)}(\xi,0) h_2^{r-1}(\xi)$$

$$+ \frac{m_1(\xi)}{\xi}\left[1-h_2(\xi)\right] \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} I_{[r\leq n]}\, {}_o r_{1j}(\xi) g_j^{(n)}(\xi,0) h_2^{r-1}(\xi)$$

where ${}_o r_{ij}(\cdot)$ is defined in (1.30), $m_1(\xi)$ in (1.52), $g_j^{(n)}(\cdot,\cdot)$ in (1.6) and:

$I_{[r\leq n]} = 1$ if $r \leq n$ and 0 if $r > n$.

<u>Proof</u>:

Let ${}_o\theta_{\nu r}(t|i)$ be the probability that at time t the server is serving the r-th customer of an arbitrary cycle of tasks, in unit ν, given that the queue has not become empty in $[0,t]$ and that at t=0 the service started with $i > 0$ customers in unit 1. Then:

(65)

$$ {}_o\theta_{1r}(t|i) = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \sum_{j_1=1}^{\infty} \int_o^t d\ {}_oQ_{ij}^{(n)}(u) \int_u^t dG_{jj_1}^{(r-1)}(u_1-u) $$

$$ \cdot \left[1 - H_1(t-u_1)\right] $$

If the queue has never become empty in [o,t], let the server be serving in the (n+1)th cycle of tasks at t, $n \geq 0$. At the end of the n-th cycle of tasks there are $j \geq 1$ customers in unit 1, given that at t=0 there were i customers in unit 1, and the n-th cycle of tasks ended between u and u+du. This probability is given by $d\,{}_oQ_{ij}^{(n)}(u)$. Starting with j customers in unit 1 at time u, there are at least (r-1) services up to time u_1 which is the last epoch of service completion before t and at the end of the (r-1)th service $j_1 \geq 1$ customers are waiting in unit 1. This probability is given by the second integral. The last factor $[1-H_1(t-u_1)]$ ensures that at time t the server is serving the r-th customer. Finally we sum over all choices of n, j, j_1, u and u_1 to obtain (65).

By a similar argument we get:

(66)

$$ {}_o\theta_{2r}(t|i) = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \sum_{\nu=j}^{\infty} I_{[r\le\nu]} \int_o^t d_oQ_{ij}^{(n)}(u)\int_u^t dG_{jo}^{(\nu)}(u_1-u) $$

$$ \cdot \int_{u_1}^t dH_2^{(r-1)}(u_2-u_1)\left[1-H_2(t-u_2)\right] $$

where the factor $I_{[r\le\nu]}$ is present because if unit 1 has ν services then unit 2 can have only at most ν services.

The Laplace transforms of (65) and (66) lead to:

(67)

$$ {}_o\theta^*_{1r}(\xi|i) = \frac{1}{\xi}\left[1-h_1(\xi)\right] \sum_{j=1}^{\infty} \sum_{j_1=1}^{\infty} {}_or_{ij}(\xi)\, g_{jj_1}^{(r-1)}(\xi) $$

(68)

$$ {}_o\theta^*_{2r}(\xi|i) = \frac{1}{\xi}\left[1-h_2(\xi)\right] \sum_{j=1}^{\infty} \sum_{\nu=j}^{\infty} I_{[r\le\nu]}\, {}_or_{ij}(\xi) g_{jo}^{(\nu)}(\xi) h_2^{r-1}(\xi) $$

The usual renewal argument gives:

(69) $$ \theta^*_{\nu r}(\xi|i) = {}_o\theta^*_{\nu r}(\xi|i) + m_1(\xi)\,{}_o\theta^*_{\nu r}(\xi|1) \ , \ \nu=1,2,. $$

Substitution of (67) and (68) in (69) proves the lemma.

<u>Lemma 2.6</u>

If $\theta^*_\nu(\xi|i)$ is the Laplace transform of $\theta_\nu(t|i)$, then:

$$(70)\qquad \theta_1^*(\xi|i) = \frac{1}{\xi}\sum_{j=1}^{\infty} {}_o r_{ij}(\xi)\left[1 - \gamma_1^j(\xi)\right]$$

$$+ \frac{1}{\xi} m_1(\xi) \sum_{j=1}^{\infty} {}_o r_{1j}(\xi)\left[1 - \gamma_1^j(\xi)\right]$$

$$(71)\qquad \theta_2^*(\xi|i) = \frac{1}{\xi}\sum_{j=1}^{\infty} {}_o r_{ij}(\xi)\left[\gamma_1^j(\xi) - \gamma_1^j(\xi, h_2(\xi))\right]$$

$$+ \frac{1}{\xi} m_1(\xi) \sum_{j=1}^{\infty} {}_o r_{1j}(\xi)\left[\gamma_1^j(\xi) - \gamma_1^j(\xi, h_2(\xi))\right]$$

where $\gamma_1(\cdot)$ and $\gamma_1(\cdot,\cdot)$ are defined in lemma 1.2.

Proof:

This is immediate upon summing over $r \geq 1$ in (63) and (64) and using lemma 1.3.

Lemma 2.7

If $1-\lambda\alpha_1-\lambda\alpha_2 > 0$, the following limiting probabilities are:

(72)

$$\text{(i)}\quad \lim_{t\to\infty} \theta_{1r}(t|i) = \lambda\alpha_1(1-\lambda\alpha_1-\lambda\alpha_2)\sum_{j=1}^{\infty}\sum_{j_1=1}^{\infty} {}_o r_{1j}(0) g_{jj_1}^{(r-1)}(0)$$

73)

$$\text{(ii)}\quad \lim_{t\to\infty} \theta_{2r}(t|i) = \lambda\alpha_2(1-\lambda\alpha_1-\lambda\alpha_2)\sum_{j=1}^{\infty}\sum_{\nu=j}^{\infty} I_{[r\leq\nu]}\, {}_o r_{1j}(0) g_{jo}^{(\nu)}(0)$$

(74)

$$\text{(iii)}\quad \lim_{t\to\infty} \theta_\nu(t|i) = \lambda\alpha_\nu, \qquad \nu=1,2,$$

Proof:

The proof follows from a Tauberian Theorem (Theorem 5, Appendix D):

$$(75) \qquad \lim_{t\to\infty} \theta_{\nu r}(t\,|\,i) = \lim_{\xi\to 0} \xi\,\theta_{\nu r}(\xi\,|\,i) \quad , \ \nu=1,2,$$

$$(76) \qquad \lim_{t\to\infty} \theta_{\nu}(t\,|\,i) = \lim_{\xi\to 0} \xi\,\theta^*_{\nu}(\xi\,|\,i) \ , \quad \nu=1,2,$$

Hence using (75) in lemma 2.5 we get (72) and (73).

Using (76) in (70) and (71) we have for $\nu=1,2$:

$$(77) \qquad \lim_{t\to\infty} \theta_{\nu}(t\,|\,i) = \frac{\lambda\alpha_\nu(1-\lambda\alpha_1-\lambda\alpha_2)}{1-\lambda\alpha_1} \sum_{j=1}^{\infty} j\ {}_0r_{1j}(0)$$

Theorem 4 in Appendix B gives:

$$\sum_{j=1}^{\infty} j\ {}_0r_{1j}(0) = 1 + \frac{\lambda\alpha_2}{1-\lambda\alpha_1-\lambda\alpha_2}$$

$$(78) \qquad = \frac{1-\lambda\alpha_1}{1-\lambda\alpha_1-\lambda\alpha_2}$$

Substitution of (78) in (77) proves (74).

Theorem 2.7

If $\theta(t\,|\,i)$ is the probability that the server is busy at time t, given that the service started at $t=0$ with $i>0$ customers in unit 1, then the Laplace transform $\theta^*(\xi\,|\,i)$ of $\theta(t\,|\,i)$ is given by:

(79)

$$\theta^*(\xi|i) = \frac{1}{\xi} \sum_{j=1}^{\infty} {}_o r_{ij}(\xi) \left[1 - \gamma_1^j(\xi, h_2(\xi))\right]$$

$$+ \frac{1}{\xi} m_1(\xi) \sum_{j=1}^{\infty} {}_o r_{1j}(\xi) \left[1 - \gamma_1^j(\xi, h_2(\xi))\right] ,$$

Further the stationary probability that the server is busy is $\lambda\alpha_1 + \lambda\alpha_2$ and hence the stationary probability that the server is idle is $1-\lambda\alpha_1-\lambda\alpha_2$.

Proof:

We have:

(80) $$\theta(t|i) = \theta_1(t|i) + \theta_2(t|i)$$

which gives:

(81) $$\theta^*(\xi|i) = \theta_1^*(\xi|i) + \theta_2^*(\xi|i)$$

Hence (79) follows from (81) and lemma 2.6.

$$\lim_{t \to \infty} \theta(t|i) = \lim_{t \to \infty} \theta_1(t|i) + \lim_{t \to \infty} \theta_2(t|i)$$

$$= \lambda\alpha_1 + \lambda\alpha_2 .$$

(by lemma 2.7)

Expected length of a cycle of tasks. Starting with $j > 0$ customers in unit 1, the expected duration of a cycle of tasks is:

$$-\frac{\partial}{\partial s} q_j(s,1)\Big]_{s=0} = -\frac{\partial}{\partial s} \gamma_1^j(s,h_2(s))\Big]_{s=0}$$

$$(82) \qquad = \frac{j(\alpha_1+\alpha_2)}{1-\lambda\alpha_1}$$

Expected sojourn time. Consider the Semi-Markov sequence $\{\xi_n, T_n, n \geq 0\}$ defined in (1.3). Let η_j be the expected sojourn time of this process in state j. Then:

$$(83a) \qquad \eta_0 = \frac{1}{\lambda} + \frac{\alpha_1 + \alpha_2}{1 - \lambda\alpha_1}$$

$$(83b) \qquad \eta_j = \frac{j(\alpha_1+\alpha_2)}{1 - \lambda\alpha_1}, \quad j \geq 1,$$

where (83a) is obtained from the fact that once the process reached the state zero, there is a negative exponential idle period with expected duration $\frac{1}{\lambda}$ and further a cycle of tasks started with a single cutomer in unit 1 whose expected value is given by (82). (83b) is obvious from (82).

Mean recurrence time. Let μ_j be the mean recurrence time of state j of the process $\{\xi_n\}$. Then:

$$\mu_0 = \frac{1}{\lambda} - \gamma'(0), \ \gamma(\cdot) \text{ defined in } (1.1),$$

$$(84) \qquad = \frac{1}{\lambda(1-\lambda\alpha_1-\lambda\alpha_2)}$$

Let $M_{ij}(t)$ be the expected number of visits to state j by the process $\{\xi_n\}$ in $(o,t]$, given that $\xi_0 = i$, and $m_{ij}(s)$ be its L.S.T. Then:

(85) $$m_{ij}(s) = m_{io}(s)\,\frac{\lambda}{\lambda+s}\,{}_{o}m_{1j}(s) \cdot {}_{o}m_{ij}(s),\ i \geq 1,\ j \geq 1$$

where ${}_{o}m_{ij}(\cdot)$ are defined in (1.30).

This is obtained from the consideration that a visit to j can occur either with or without an intermediate visit to the state 0 [Neuts (1969)].

From (85) we get for $j \geq 1$:

$$\mu_j^{-1} = \lim_{s \to 0} s\, m_{ij}(s)$$

$$= {}_{o}m_{1j}(0) \lim_{s \to 0} s\, m_{io}(s)$$

(86) $$= {}_{o}m_{1j}(0)\, \mu_o^{-1}$$

If $p_{ij}(t)$ is the probability that the Semi-Markov process is in state j, given that it started in state i at t=0, and $p_j^* = \lim_{t \to \infty} p_{ij}(t)$ then:

(87) $$p_j^* = \frac{\eta_j}{\mu_j}\ ,\ j \geq 0\ ,$$

Substitution of (83), (84) and (86) in (87) yields:

$$p_o^* = (1 + \lambda\alpha_2)\,(1 - \lambda\alpha_1 - \lambda\alpha_2)\ |\ (1 - \lambda\alpha_1)$$

and

$$p_j^* = \{\lambda(\alpha_1+\alpha_2)(1-\lambda\alpha_1-\alpha_2)j\ {}_{o}m_{1j}(0)\}\,|(1-\lambda\alpha_1),\quad j \geq 1,$$

7. Generalizations

The Tandem Queue With More Than Two Units

Let us consider $m(\geq 2)$ service units. The input to unit 1 is a Poisson process of density λ, and the input to the $(r+1)$th unit is the output from the r-th unit, $r=1,2,\ldots,m-1$. After getting service in the m-th unit the customers depart from the whole system.

At $t=0$ a single server starts serving in unit 1. He switches from unit 1 either by a zero switching rule or by a non-zero switching rule, while he always observes a zero switch rule in all other units $2,\ldots,m$. In all the units the customers are served by the order of their arrivals and the server is busy as long as there is at least one customer in the whole system. Service times are assumed to be mutually independent positive random variables and independent of arrival times.

Each cycle of tasks consists of m tasks, task-1,..., task-m, and each busy period consists of a random number of such cycles.

Let $H_1(\cdot),\ldots,H_m(\cdot)$ be the service time distributions in unit 1,...,unit m respectively. The distributions of busy period and virtual waiting time and queue length are obtained by replacing $H_2(\cdot)$ by $H_2(\cdot)*H_3(\cdot)*\ldots*H_m(\cdot)$ in the results of two units. The corresponding moments are obtained by replacing α_2 by $\sum_{i=2}^{m} \alpha_i$ and β_2 by $\sum_{i=2}^{m} \beta_i + 2 \sum_{i=2}^{m-1} \sum_{j=i+1}^{m} \alpha_i \alpha_j$.

where $\alpha_\nu = \int_0^\infty x dH_\nu(x)$ and $\beta_\nu = \int_0^\infty x^2 dH_\nu(x)$, $\nu=1,\ldots,m$

Infinite Tandem Queues. Suppose that the number of service units m is infinite. Let S_ν be the service time of a customer in unit ν, $\nu=1,2,\ldots$. Then $S_1, S_2, \ldots$ are independent random variables with distribution functions $H_1(\cdot)$, $H_2(\cdot),\ldots$

Theorem 2.7. Convergence Theorem

(a) If $\alpha = \sum_{\nu=1}^{\infty} \alpha_\nu$ and $\beta = \sum_{\nu=1}^{\infty} \beta_\nu < \infty$ then the distribution $G_m(\cdot)$ of the service time of a customer in the first m units, $S_1 + \ldots + S_m$, converges to a probability distribution $G_*(\cdot)$ with first and second moments α and β respectively.

(b) The total service time $\sum_{\nu=1}^{\infty} S_\nu$ of a customer converges in law if and only if for a fixed $c > o$ the three series

(i) $\sum_{n=1}^{\infty} \int_c^\infty dH_n(x)$, (ii) $\sum_{n=1}^{\infty} \alpha_n^{(c)}$ and (iii) $\sum_{n=1}^{\infty} \beta_n^{(c)}$ converge,

where $\alpha_n^{(c)} = \int_0^c x\, dH_n(x)$ and $\beta_n^{(c)} = \int_0^c x^2\, dH_n(x)$.

For the proof of this theorem we refer to Feller (1966).

If the service time distributions are negative exponential, $H_\nu(x) = 1-e^{-\mu_\nu x}$, $\nu=1,2,\ldots$, then by the convergence theorem the distribution of $S_1+\ldots+S_m$ converges to a probability distribution $G_*(\cdot)$ if $\sum_{\nu=1}^{\infty} \frac{1}{\mu_\nu} < \infty$. $G_*(t)$ gives the probability that a customer will be served in infinitely many units before epoch t.

Under the conditions of convergence we have:

$$P\left\{\sum_{\nu=1}^{\infty} S_\nu \le x\right\} = \mathop{\pi}_{\nu=1}^{\infty} * H_\nu(x)$$

which is the convolution of $H_1(\cdot)$, $H_2(\cdot)$,....
Hence the distributions of busy period, virtual waiting time and queue length are obtained by replacing $H_2(\cdot)$ by $\mathop{\pi}_{\nu=2}^{\infty} * H_\nu(\cdot)$ in the results of two units.

Equilibrium Conditions of the Infinite Tandem Queues. Under the conditions of convergence of the total service time of a customer, the queue will attain its equilibrium if $1-\lambda \sum_{\nu=1}^{\infty} \alpha_\nu > 0$. This follows from the results of an $M|G|1$ queue with service time distribution $\mathop{\pi}_{\nu=1}^{\infty} * H_\nu(x)$.

The Tandem Queue with Balking

Consider the tandem queue with two units. Let p be the probability that a customer joins the queue in unit 2 and 1-p the probability that he leaves the system after getting service in unit 1. The distribution of busy period, virtual waiting time and queue length of this model can easily be studied from the following considerations: We assume that all the customers after getting service in unit 1 go through unit 2 and get a non-zero service there with probability p and a zero service with probability 1-p. The distribution of service time of a customer entering unit 2 is:

$$(88) \qquad p\, H_2(\cdot) + (1-p)\, U(\cdot)$$

Hence the distributions of busy period, virtual waiting time, etc. can be obtained from the non-balking case by replacing $H_2(\cdot)$ by (88). To get the moments, α_2 is replaced by $p\,\alpha_2$ and β_2 by $p\,\beta_2$.

In the case of the tandem queue with m service units, let p_ν be the probability that a customer joins the queue in unit $\nu+1$ and $1-p_\nu$ the probability that he leaves the system after getting service in unit ν, $\nu=1,2,\ldots,m-1$. Here also we assume that each customer after getting service in unit 1 goes through all the remaining (m-1) units and gets a non-zero service in unit ν with probability $(\prod_{i=1}^{\nu-1} p_i)$ and a zero service with probability $\left[1 - (\prod_{i=1}^{\nu-1} p_i)\right]$, $\nu=2,\ldots,m$. The distribution of service time of a customer in unit ν is:

$$(89) \qquad (\prod_{i=1}^{\nu-1} p_i)\, H_\nu(\cdot) + \left[1 - (\prod_{i=1}^{\nu-1} p_i)\right] U(\cdot)$$

Hence the different distributions of interest can be obtained from the non-balking case with m service units by replacing $H_\nu(\cdot)$ by (89), $\nu=2,\ldots,m$.

8. Applications

The tandem models considered in Chapters I and II can be viewed as a modified alternating priority model. In the alternating priority model [Avi-Itzhak, Maxwell and Miller (1965),

Neuts and Yadin (1968), Takács (1968)] customers arrive at two service units, unit 1 and unit 2, in accordance with Poisson process of densities λ_1 and λ_2. A single server attends to two units alternately according to zero switch rule and serves the customers in the order of their arrivals. In this alternating priority model suppose that the input to unit 2 is stored there as long as the server is serving in unit 1. Once the server started serving in unit 2 the input to it is shut off and stored in unit 1 until he switches back to unit 1. As soon as the server switches back to unit 1 the stored input of unit 2 is released from unit 1 to unit 2. This modification is reasonable in cases where the arrival of a customer in unit 2 causes service interruption there or in cases where only those customers of unit 2 who have arrived during the service time of the customers of unit 1 have priority over the customers arriving in unit 1 thereafter.

The analysis of this modified alternating priority model can be easily deduced from our tandem model: Customers arrive at a service system in accordance with a Poisson process of density λ. Independently of others an arriving customer is of type 1 with probability p_1 or of type 2 with probability p_2, where $p_1+p_2 = 1$, $\lambda_1 = \lambda p_1$, $\lambda_2 = \lambda p_2$. All the arriving customers pass through both the units 1 and 2. A type 1 customer receives a non-zero service in unit 1 and zero service in unit 2, while a type 2 customer receives a zero service in unit 1 and a non-zero

service in unit 2. The service time distribution in unit 1 of an arriving customer is $H_1(\cdot)$ with probability p_1 and $U(\cdot)$ with probability p_2, while his service time distribution in unit 2 is $H_2(\cdot)$ with probability p_2 and $U(\cdot)$ with probability p_1, where $U(\cdot)$ is the unit distribution. Hence in our analysis in Chapters I and II we replace $H_1(x)$ by $p_1H_1(x) + p_2U(x)$ and $H_2(x)$ by $p_2H_2(x) + p_1U(x)$.

CHAPTER III

ALTERNATING PRIORITY QUEUES WITH NON-ZERO SWITCHING

1. Concepts and Definitions

This chapter discusses a queueing model in which a single server serves two units 1 and 2; the input processes to these are independent Poisson processes of rates λ_1 and λ_2 respectively. The server attends the two units alternately according to a non-zero switching rule. He continues to serve in unit ν until he has given k_ν services without interruption there or until the queue becomes empty whichever comes first. k_ν $\nu=1,2$, are positive integers, which are called the switching parameters. The alternating priority queues with zero-switching ($k_1=k_2=\infty$) have been studied by several authors: Avi-Itzhak, Maxwell and Miller (1965), Neuts and Yadin (1968), Takács (1968).

It is assumed that at both units customers are served in the order of their arrivals. The service times are mutually independent positive random variables; independent of the arrival times. Denote by $H_1(\cdot)$ and $H_2(\cdot)$ the distribution functions of service times in units 1 and 2 respectively.

We use the following notation:

$$h_\nu(s) = \int_0^\infty e^{-sx} dH_\nu(x), \quad \nu=1,2, \; R(s) \geq 0,$$

$$\alpha_\nu = \int_0^\infty x \, dH_\nu(x), \quad \nu=1,2.$$

2. Distribution of Busy Period

We recall that a task is the time interval spent without interruption in a unit. A task in unit ν is referred to as a ν-task. It consists of atmost k_ν consecutive services $\nu=1,2$.

Suppose that at t=0 the server starts serving in a unit. The time required for both the units to become empty simultaneously for the first time is called a busy period. If the busy period starts with the service of a customer in unit ν then the corresponding busy period is called a ν-busy period (or busy period of type ν), $\nu=1,2$. Let $\kappa_\nu(\cdot)$ denote the distribution function of type ν busy period, $\nu=1,2$.

The system becomes idle when both the units are empty. The idle period has a negative exponential distribution with parameter $\lambda_1 + \lambda_2$. After an idle period a new busy period starts in the unit in which a customer arrives first.

Remark:

As in Neuts and Yadin (1968), if the unit to which the server switches is empty then we assume that he instantaneously

completes a task of duration of zero there and switches back to the other unit.

Since the distribution of busy period does not depend on the switching rule, [Welch (1965)], it follows from Neuts and Yadin (1968) that:

<u>Theorem 3.1</u>

If $\theta_1(s)$ and $\theta_2(s)$ are the L.S.T of $\varkappa_1(\cdot)$ and $\varkappa_2(\cdot)$ respectively then:

(i) For every s wtih $R(s) > 0$, the pair $\theta_1(s)$ and $\theta_2(s)$ is the unique solution to the following system of equations:

(1) (a)

$$z_1 = h_1(s+\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2), \quad z_2 = h_2(s+\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)$$

(2) (b) $z_1 = \gamma_1(s+\lambda_2-\lambda_2 z_2)$, $z_2 = \gamma_2(s+\lambda_1-\lambda_1 z_1)$

in the region $|z_1| \leq 1$, $|z_2| \leq 1$, where $\gamma_\nu(\cdot)$ is the L.S.T of the distribution of busy period of an $M|G|1$ queue with input rate λ_ν and service time distribution $H_\nu(\cdot)$, $\nu=1,2$.

(3)(ii) $\theta_1(0+) = \theta_2(0+) = 1$ if and only if $1-\lambda\alpha_1-\lambda\alpha_2 \geq 0$

(iii) If $1-\lambda\alpha_1-\lambda\alpha_2 > 0$ then the means of $\varkappa_1(\cdot)$ and $\varkappa_2(\cdot)$ are given by:

$$(4) \qquad -\theta_1'(0+) = \frac{\alpha_1}{1-\lambda\alpha_1-\lambda\alpha_2}, \quad -\theta_2'(0+) = \frac{\alpha_2}{1-\lambda\alpha_1-\lambda\alpha_2}$$

3. The Basic Imbedded Semi-Markov Process and its Transition Probabilities

We suppose that at t=0 there are $i_1 \geq 1$ customers in unit 1 and $i_2 \geq 0$ customers in unit 2. Furthermore a customer in unit 1 is just beginning service. We may also start with other initial conditions.

Let us define the sequence of random variables $t_o, t_1, \ldots$, where $t_o=0$ and t_n is the duration of the n-th task. The odd numbered variables $t_1, t_3, \ldots$ are the durations of tasks in unit 1 and the even numbered variables $t_2, t_4, \ldots$ are the durations of tasks in unit 2. Let $\underset{\sim}{\xi}_n = (\xi_n^{(1)}, \xi_n^{(2)})$ be the number of customers in the system (unit 1, unit 2) at the end of the n-th task, $n \geq 1$ and $\underset{\sim}{\xi}_o = (i_1, i_2)$. Further let ζ_n be a random variable which takes values 1 and 2 depending on whether the (n+1)th task is a 1-task or a 2-task, $n \geq 1$, $\zeta_o=1$. It then follows from the regenerative properties of the input and service processes that the quadrivariate sequence of random variables:

$$\left\{\zeta_n,\ \xi_n^{(1)},\ \xi_n^{(2)},\ t_n,\ n \geq 0\right\} \tag{5}$$

is a Semi-Markov sequence with state space:

$$\left\{1,2\right\} \times \left\{0,1,\ldots\right\} \times \left\{0,1,\ldots\right\}$$

To study the transition probabilities of the semi-Markov sequence defined in (5) we define the auxiliary probability functions ${}_1G_{ij}^{(n)}(x)$ and ${}_2G_{ij}^{(n)}(x)$ as:

(6a) $$ {}_{\nu}G_{ij}^{(o)}(x) = \delta_{ij}\, U(x) \quad , \ \nu=1,2, $$

and for $n \geq 1$, ${}_{\nu}G_{ij}^{(n)}(x)$ is the probability that, in an M|G|1 queue of input rate λ_{ν} and service time distribution $H_{\nu}(\cdot)$ the initial busy period involves at least n services, that the n-th service is completed before time x and that at the end of the n-th service there are j customers waiting, given that there were i customers initially, ν=1,2.

(6b) $$ {}_{\nu}G_{ij}^{(1)}(x) = \int_{0}^{x} e^{-\lambda_{\nu} y} \frac{(\lambda_{\nu} y)^{j-i+1}}{(j-i+1)!}\, d\, H_{\nu}(y), \ \nu=1,2, $$

(6c)
$$ {}_{\nu}G_{ij}^{(n+1)}(x) = \sum_{r=1}^{j+1} \int_{0}^{x} {}_{\nu}G_{ir}^{(n)}(x-y) e^{-\lambda_{\nu} y} \frac{(\lambda_{\nu} y)^{j-r+1}}{(j-r+1)!}\, dH_{\nu}(y), \ n \geq 1, \quad \nu=1,2, $$

Let ${}_{\nu}g_{ij}^{(n)}(s)$ be the L.S.T. of ${}_{\nu}G_{ij}^{(n)}(x)$ and

(7) $$ {}_{\nu}g_{i}^{(n)}(s,z) = \sum_{j=0}^{\infty} {}_{\nu}g_{ij}^{(n)}(s)\, z^{j}, \ |z| \leq 1, \ \nu=1,2, $$

Then for ν=1,2:
$$ {}_{\nu}g_{i}^{(0)}(s,z) = z^{i} $$

(8) $$ {}_{\nu}g_{i}^{(1)}(s,z) = z^{i-1}\, h_{\nu}(s+\lambda_{\nu}-\lambda_{\nu} z) $$

$$ {}_{\nu}g_{i}^{(n+1)}(s,z) = \frac{h_{\nu}(s+\lambda_{\nu}-\lambda_{\nu} z)}{z} \left[{}_{\nu}g_{i}^{(n)}(s,z) - {}_{\nu}g_{i}^{(n)}(s,0) \right], \quad n \geq 1, $$

The results analogous to lemma 1.1 through lemma 1.5 are easily seen to be satisfied by the probability functions ${}_{\nu}G_{ij}^{(n)}(x)$.

Let us denote:

(9) $\underset{\sim}{i} = (i_1,i_2)$, $\underset{\sim}{j} = (j_1,j_2)$, $\underset{\sim}{0} = (0,0)$, $\underset{\sim}{z} = (z_1,z_2)$

Define the transition probability functions:

(10) $$Q_1(\underset{\sim}{i},\underset{\sim}{j},x) = P\left\{\zeta_n=2, \xi_n^{(1)}=j_1, \xi_n^{(2)}=j_2, T_n \le x \;\middle|\; \zeta_{n-1}=1, \xi_{n-1}^{(1)}=i_1, \xi_{n-1}^{(2)}=i_2\right\}$$

(11) $$Q_2(\underset{\sim}{i},\underset{\sim}{j},x) = P\left\{\zeta_n=1, \xi_n^{(1)}=j_1, \xi_n^{(2)}=j_2, T_n \le x \;\middle|\; \zeta_{n-1}=2, \xi_{n-1}^{(1)}=i_1, \xi_{n-1}^{(2)}=i_2\right\}$$

We have:

(12)
$$Q_1(\underset{\sim}{i},\underset{\sim}{j},x) = \int_0^x d_1G_{i_1j_1}^{(k_1)}(u)e^{-\lambda_2 u}\frac{(\lambda_2 u)^{j_2-i_2}}{(j_2-i_2)!}\quad , \text{ if } j_1 \ge 1,\ i_1 \ge 1\ \ j_2 \ge i_2 \ge 0,$$

$$= \sum_{r=i_1}^{k_1}\int_0^x d_1G_{i_1 0}^{(r)}(u)\, e^{-\lambda_2 u}\frac{(\lambda_2 u)^{j_2-i_2}}{(j_2-i_2)!}\ , \text{ if } j_1 = 0\ \ j_2 \ge i_2 \ge 0,\ i_1 \ge 1,$$

$= \delta_{i_2j_2}\, U(x)$, if $i_1 = j_1 = 0$, $i_2 \ge 1$

$= 0$ for all other choices of the indices

except for $i_1 = i_2 = 0$,

(13)

$$Q_2(\underset{\sim}{i},\underset{\sim}{j},x) = \int_0^x d_2G_{i_2j_2}^{(k_2)}(u)\, e^{-\lambda_1 u}\, \frac{(\lambda_1 u)^{j_1-i_1}}{(j_1-i_1)!}\ ,$$

$$\text{if } j_2 \geq 1,\ j_1 \geq i_1 \geq 0,\ i_2 \geq 1$$

$$= \sum_{r=i_2}^{k_2} \int_0^x d_2G_{i_2 o}^{(r)}(u)\, e^{-\lambda_1 u}\, \frac{(\lambda_1 u)^{j_1-i_1}}{(j_1-i_1)!}\ ,$$

$$\text{if } j_2 = 0,\ j_1 \geq i_1 \geq 0,\ i_2 \geq 1$$

$$= \delta_{i_1 j_1}\, U(x)\ ,\ \text{ if } i_2 = j_2 = 0,\ i_1 \geq 1$$

$$= 0 \text{ for all other indices except } i_1 = i_2 = 0$$

For $i_1 = 0 = i_2$:

(14) $$Q_1(\underset{\sim}{0},\underset{\sim}{j},x) = \int_0^x Q_1(1,0;\ \underset{\sim}{j},\ x-u)\, e^{-(\lambda_1+\lambda_2)u}\, \lambda_1\, du\ ,$$

(15) $$Q_2(\underset{\sim}{0},\underset{\sim}{j},x) = \int_0^x Q_2(0,1;\underset{\sim}{j},x-u)\, e^{-(\lambda_1+\lambda_2)u}\, \lambda_2\, du\ ,$$

Let $q_\nu(\underset{\sim}{i},\underset{\sim}{j},s)$ be the L.S.T. of $Q_\nu(\underset{\sim}{i},\underset{\sim}{j},x)$ and

(16)

$$q_\nu^*(\underset{\sim}{i},\underset{\sim}{z},s) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} q_\nu(\underset{\sim}{i},\underset{\sim}{j},s)\, z_1^{j_1}\, z_2^{j_2}\ ,\ |z_\nu| \leq\ ,\ \nu=1,2,$$

From (12) to (15) we get:

$$(17)\quad q_1^*(\underset{\sim}{i},\underset{\sim}{z},s) = z_2^{i_2}\Big\{{}_1g_{i_1}^{(k_1)}(s+\lambda_2-\lambda_2 z_2, z_1)$$

$$+ \sum_{r=i_1}^{k_1-1} {}_1g_{i_1}(s+\lambda_2-\lambda_2 z_2, 0)\Big\},\ \text{if } i_1 \geq 1,$$

$$= z_2^{i_2},\ \text{if } i_1 = 0,\ i_2 \geq 1$$

$$(18)\quad q_2^*(\underset{\sim}{i},\underset{\sim}{z},s) = z_1^{i_1}\Big\{{}_2g_{i_2}^{(k_2)}(s+\lambda_1-\lambda_1 z_1, z_2)$$

$$+ \sum_{r=i_2}^{k_2-1} {}_2g_{i_2}(s+\lambda_1-\lambda_1 z_1, 0)\Big\},\ \text{if } i_2 \geq 1,$$

$$= z_1^{i_1},\ \text{if } i_2 = 0,\ i_1 \geq 1$$

$$(19)\quad q_1^*(\underset{\sim}{0},\underset{\sim}{z},s) = \frac{\lambda_1}{\lambda_1+\lambda_2+s}\, q_1^*(1,0;\ \underset{\sim}{z},s)$$

$$(20)\quad q_2^*(\underset{\sim}{0},\underset{\sim}{z},s) = \frac{\lambda_2}{\lambda_1+\lambda_2+s}\, q_2^*(0,1;\ \underset{\sim}{z},s)$$

Let $R_n(\underset{\sim}{i},\underset{\sim}{j},x)$ be the probability that a busy period lasts for at least n tasks, that the n-th task ends not later than time x and that at the end of the n-th task $\underset{\sim}{j} = (j_1, j_2)$ customers are waiting in units 1 and 2 respectively, given that the service started with $\underset{\sim}{i}$ customers, $i_1 \geq 1$, in unit 1 at t=0. Then:

$$(21a)\qquad R_1(\underset{\sim}{i},\underset{\sim}{j},x) = Q_1(\underset{\sim}{i},\underset{\sim}{j},x),\ i_1 \geq 1,$$

(21b) $$R_{2n+1}(\underset{\sim}{i},\underset{\sim}{j},x) = \sum_{\nu_1=1}^{\infty} \sum_{\nu_2=0}^{\infty} \int_0^x R_{2n}(\underset{\sim}{i},\underset{\sim}{\nu},x-u)$$

$$\cdot\, dQ_1(\underset{\sim}{\nu},\underset{\sim}{j},u), \quad n \geq 1, \quad i_1 \geq 1,$$

(21c) $$R_{2n}(\underset{\sim}{i},\underset{\sim}{j},x) = \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=1}^{\infty} \int_0^x R_{2n-1}(\underset{\sim}{i},\underset{\sim}{\nu},x-u)$$

$$dQ_2(\underset{\sim}{\nu},\underset{\sim}{j},u), \quad n \geq 1, \; i_1 \geq 1,$$

Further let $r_n(\underset{\sim}{i},\underset{\sim}{j},s)$ be the L.S.T. of $R_n(\underset{\sim}{i},\underset{\sim}{j},x)$ and

(22) $$r_n^*(\underset{\sim}{i},\underset{\sim}{z},s) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} r_n(\underset{\sim}{i},\underset{\sim}{j},s)\, z_1^{j_1}\, z_2^{j_2} ,$$

$$|z_\nu| \leq 1, \; \nu = 1,2, \quad i_1 \geq 1,$$

so that (21) gives:

(23a) $$r_1(\underset{\sim}{i},\underset{\sim}{j},s) = q_1(\underset{\sim}{i},\underset{\sim}{j},s) ,$$

(23b) $$r_{2n+1}(\underset{\sim}{i},\underset{\sim}{j},s) = \sum_{\nu_1=1}^{\infty} \sum_{\nu_2=0}^{\infty} r_{2n}(\underset{\sim}{i},\underset{\sim}{\nu},s)\, q_1(\underset{\sim}{\nu},\underset{\sim}{j},s),$$

$$n \geq 1, \; i_1 \geq 1,$$

(23c) $$r_{2n}(\underset{\sim}{i},\underset{\sim}{j},s) = \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=1}^{\infty} r_{2n-1}(\underset{\sim}{i},\underset{\sim}{\nu},s) q_2(\underset{\sim}{\nu},\underset{\sim}{j},s),$$

$$n \geq 1, \; i_1 \geq 1,$$

(24a) $$r_1^*(\underset{\sim}{i},\underset{\sim}{z},s) = q_1(\underset{\sim}{i},\underset{\sim}{z},s) \quad , \ i_1 \geq 1,$$

(24b) $$r_{2n+1}^*(\underset{\sim}{i},\underset{\sim}{z},s) = \sum_{\nu_1=1}^{\infty} \sum_{\nu_2=0}^{\infty} r_{2n}(\underset{\sim}{i},\underset{\sim}{\nu},s)q_1(\underset{\sim}{\nu},\underset{\sim}{z},s) \ ,$$

$$n \geq 1, \ i_1 \geq 1,$$

(24c) $$r_{2n}^*(\underset{\sim}{i},\underset{\sim}{z},s) = \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=1}^{\infty} r_{2n-1}(\underset{\sim}{i},\underset{\sim}{\nu},s)q_2(\underset{\sim}{\nu},\underset{\sim}{z},s) \ ,$$

$$n \geq 1, \ i_1 \geq 1,$$

Analogously we define $\widetilde{R}_n(\underset{\sim}{i},\underset{\sim}{j},x)$ as the probability that a busy period lasts for at least n tasks, that the n-th task ends not later than time x and that at the end of the n-th task $\underset{\sim}{j}$ customers are waiting, given that the service started with $\underset{\sim}{i}$ customers, $i_2 \geq 1$, in <u>unit 2</u> at t=0. Recalling (21):

(25a) $$\widetilde{R}_1(\underset{\sim}{i},\underset{\sim}{j},x) = Q_2(\underset{\sim}{i},\underset{\sim}{j},x), \quad i_2 \geq 1,$$

(25b) $$\widetilde{R}_{2n+1}(\underset{\sim}{i},\underset{\sim}{j},x) = \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=1}^{\infty} \int_0^x \widetilde{R}_{2n}(\underset{\sim}{i},\underset{\sim}{\nu},x-u)dQ_2(\underset{\sim}{\nu},\underset{\sim}{j},u) \ ,$$

$$n \geq 1, \ i_2 \geq 1,$$

(25c) $$\widetilde{R}_{2n}(\underset{\sim}{i},\underset{\sim}{j},x)= \sum_{\nu_1=1}^{\infty} \sum_{\nu_2=0}^{\infty} \int_0^x \widetilde{R}_{2n-1}(\underset{\sim}{i},\underset{\sim}{\nu},x-u)dQ_1(\underset{\sim}{\nu},\underset{\sim}{j},u)$$

$$n \geq 1, \ i_2 \geq 1,$$

Similar to (23) and (24) we get the recurrence relations of the transforms $\widetilde{r}_n(\underset{\sim}{i},\underset{\sim}{j},s)$ and $\widetilde{r}_n^*(\underset{\sim}{i},\underset{\sim}{z},s)$ of $\widetilde{R}_n(\underset{\sim}{i},\underset{\sim}{j},x)$. We see further that:

$$(26) \qquad r_n(\underset{\sim}{i},\underset{\sim}{j},s) = \sum_{\nu_1=0}^{\infty}\sum_{\nu_2=1}^{\infty} q_1(\underset{\sim}{i},\underset{\sim}{\nu},s)\tilde{r}_{n-1}(\underset{\sim}{\nu},\underset{\sim}{j},s),$$

and

$$(27) \qquad \tilde{r}_n(\underset{\sim}{i},\underset{\sim}{j},s) = \sum_{\nu_1=1}^{\infty}\sum_{\nu_2=0}^{\infty} q_2(\underset{\sim}{i},\underset{\sim}{\nu},s)r_{n-1}(\underset{\sim}{\nu},\underset{\sim}{j},s),$$

4. The Queue length Process

Let us denote by $\xi_1(t)$ and $\xi_2(t)$ the numbers of customers who still require some service in units 1 and 2 respectively at time t. As in Neuts and Yadin (1968) we further denote:

$$(28) \qquad \pi_1(\underset{\sim}{i},\underset{\sim}{j},t) = P_1\left\{\xi_1(t)=j_1,\xi_2(t)=j_2 \,\middle|\, \xi_1(0)=i_1,\xi_2(0)=i_2\right\}$$

and

$$(29) \qquad \pi_2(\underset{\sim}{i},\underset{\sim}{j},t) = P_2\left\{\xi_1(t)=j_1,\xi_2(t)=j_2 \,\middle|\, \xi_1(0)=i_1,\xi_2(0)=i_2\right\}$$

where the subscripts 1 or 2 denotes that at t the server is in unit 1 or 2 respectively.

Let for $\nu=1,2$, $\phi_\nu(\underset{\sim}{i},\underset{\sim}{j},t)$ be the probability that at t there are $\underset{\sim}{j} = (j_1,j_2)$ customers in units 1 and 2 respectively, that the queue is never empty in $(0,t]$ and that the original task has not ended, given that the service started in unit ν at $t=0$ with $\underset{\sim}{i}$ customers.

Further let $\psi^*_\nu(\underset{\sim}{i},\underset{\sim}{j},\xi)$ be the Laplace transform of $\psi_\nu(\underset{\sim}{i},\underset{\sim}{j},t)$ and

$$(30)\qquad \psi^{**}_\nu(\underset{\sim}{i},\underset{\sim}{z},\xi) = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} \psi^*_\nu(\underset{\sim}{i},\underset{\sim}{j},\xi) z_1^{j_1} z_2^{j_2},$$

$$|z_\nu| \le 1,\ \nu=1,2,$$

Lemma 3.1

The transforms $\psi^{**}_\nu(\underset{\sim}{i},\underset{\sim}{z},\xi)$ of $\psi_\nu(\underset{\sim}{i},\underset{\sim}{j},t)$ are given by:

(31)

$$\begin{aligned}\psi^{**}_1(\underset{\sim}{i},\underset{\sim}{z},\xi) = z_2^{i_2}\Big\{&(\xi+\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)\\ &\cdot\left[z_1-h_1(\xi+\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)\right]\Big\}^{-1}\\ &\cdot\left[1-h_1(\xi+\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)\right]\\ &\cdot\Big\{z_1^{i_1+1}\left[1-z_1^{-k_1}h_1^{k_1}(\xi+\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)\right]\\ &-\sum_{\nu=1}^{k_1-1} z_1\left[1-z_1^{\nu-k_1}h_1^{k_1-\nu}(\xi+\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)\right]\\ &\cdot {}_1g_{i_1}^{(\nu)}(\xi+\lambda_2-\lambda_2 z_2,0)\Big\},\quad i_1\ge 1,\end{aligned}$$

$$\begin{aligned}(32)\qquad \psi^{**}_2(\underset{\sim}{i},\underset{\sim}{z},\xi) = z_1^{i_1}\Big\{&(\xi+\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)\left[z_2-h_2(\xi+\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)\right]\Big\}^{-1}\\ &\cdot\left[1-h_2(\xi+\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)\right]\\ &\cdot\Big\{z_2^{i_2-1}\left[1-z_2^{-k_2}h_2^{k_2}(\xi+\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)\right]\\ &-\sum_{\nu=1}^{k_2-1} z_2\left[1-z_2^{\nu-k_2}h_2^{k_2-\nu}(\xi+\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)\right]\\ &\cdot {}_2g_{i_2}^{(\nu)}(\xi+\lambda_1-\lambda_1 z_1,0)\Big\},\quad i_2\ge 1,\end{aligned}$$

Proof:

We have:

(33)

$$\psi_1(\underset{\sim}{i},\underset{\sim}{j},t) = \sum_{r=0}^{k_1-1} \sum_{\nu=1}^{j_1} \int_0^t d_1G_{i_1\nu}^{(r)}(u)e^{-\lambda_1(t-u)} \frac{[\lambda_1(t-u)]^{j_1-\nu}}{(j_1-\nu)!} \cdot e^{-\lambda_2 t} \frac{(\lambda_2 t)^{j_2-i_2}}{(j_2-i_2)!}\left[1-H_1(t-u)\right], i_1 \geq 1,$$

(34)

$$\psi_2(\underset{\sim}{i},\underset{\sim}{j},t) = \sum_{r=0}^{k_2-1} \sum_{\nu=1}^{j_2} \int_0^t d_2G_{i_2\nu}^{(r)}(u)e^{-\lambda_2(t-u)} \frac{[\lambda_2(t-u)]^{j_2-\nu_2}}{(j_2-\nu_2)!} \cdot e^{-\lambda_1 t} \frac{(\lambda_1 t)^{j_1-i_1}}{(j_1-i_1)!}\left[1-H_2(t-u)\right], i_2 \geq 1,$$

The probabilistic arguments for these are similar to those in Chapter I. Upon taking transforms in (33) and (34) we obtain:

(35)

$$\psi_1^{**}(\underset{\sim}{i},\underset{\sim}{z},\xi) = z_2^{i_2}\left[1-h_1(\xi+\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)\right](\xi+\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)^{-1} \cdot \sum_{r=0}^{k_1-1}\left\{{}_1g_{i_1}^{(r)}(\xi+\lambda_2-\lambda_2 z_2,z_1)-{}_1g_{i_1}^{(r)}(\xi+\lambda_2-\lambda_2 z_2,0)\right\}, i_1 \geq 1,$$

(36)

$$\psi_2^{**}(\underset{\sim}{i},\underset{\sim}{z},\xi) = z_1^{i_1}\left[1-h_2(\xi+\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)\right](\xi+\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)^{-1} \cdot \sum_{r=0}^{k_2-1}\left\{{}_2g_{i_2}^{(r)}(\xi+\lambda_1-\lambda_1 z_1,z_2) - {}_2g_{i_2}^{(r)}(\xi+\lambda_1-\lambda_1 z_1,0)\right\}$$

The lemma follows now by simplifying (35) and (36) with the help of (1.8b).

For r=1,2; ν=1,2, let $\phi_{r\nu}(\underset{\sim}{i},\underset{\sim}{j},t)$ be the probability that at time t there are $\underset{\sim}{j} = (j_1,j_2)$ customers in units 1 and 2 respectively, that the queue is never empty in (0,t] and that the server is serving in unit ν, given that the service started in unit r at t=0 with $\underset{\sim}{i}$ customers.

In terms of the functions ψ_ν and R_n we have:

$$(37)\qquad \phi_{11}(\underset{\sim}{i},\underset{\sim}{j},t) = \psi_1(\underset{\sim}{i},\underset{\sim}{j},t) + \sum_{\nu_1=1}^{\infty}\sum_{\nu_2=0}^{\infty}\sum_{n=1}^{\infty}\int_0^t \psi_1(\underset{\sim}{\nu},\underset{\sim}{j},t-u)\, dR_{2n}(\underset{\sim}{i},\underset{\sim}{\nu},u), \quad i_1 \geq 1$$

$$(38)\qquad \phi_{12}(\underset{\sim}{i},\underset{\sim}{j},t) = \sum_{\nu_1=0}^{\infty}\sum_{\nu_2=1}^{\infty}\sum_{n=0}^{\infty}\int_0^t \psi_2(\underset{\sim}{\nu},\underset{\sim}{j},t-u)\, dR_{2n+1}(\underset{\sim}{i},\underset{\sim}{\nu},u)\ , \quad i_1 \geq 1,$$

$$(39)\qquad \phi_{21}(\underset{\sim}{i},\underset{\sim}{j},t) = \sum_{\nu_1=1}^{\infty}\sum_{\nu_2=0}^{\infty}\sum_{n=0}^{\infty}\int_0^t \psi_1(\underset{\sim}{\nu},\underset{\sim}{j},t-u)\, d\tilde{R}_{2n+1}(\underset{\sim}{i},\underset{\sim}{\nu},u), \quad i_2 \geq 1,$$

$$(40)\qquad \phi_{22}(\underset{\sim}{i},\underset{\sim}{j},t) = \psi_2(\underset{\sim}{i},\underset{\sim}{j},t) + \sum_{\nu_1=0}^{\infty}\sum_{\nu_2=1}^{\infty}\sum_{n=1}^{\infty}\int_0^t \psi_2(\underset{\sim}{\nu},\underset{\sim}{j},t-u)\, d\tilde{R}_{2n}(\underset{\sim}{i},\underset{\sim}{\nu},u), \quad i_2 \geq 1,$$

If $\phi^*_{r\nu}(\underset{\sim}{i},\underset{\sim}{j},\xi)$ is the Laplace transform of $\phi_{r\nu}(\underset{\sim}{i},\underset{\sim}{j},t)$

and

(41) $$\phi^{**}_{r\nu}(\underset{\sim}{i},\underset{\sim}{z},\xi) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \phi^*_{r\nu}(\underset{\sim}{i},\underset{\sim}{j},\xi) z_1^{j_1} z_2^{j_2} ,$$

$$|z_\nu| \leq 1, \; r=1,2, \; \nu=1,2,$$

then formulae (37) to (40) give:

(42) $$\phi^{**}_{11}(\underset{\sim}{i},\underset{\sim}{z},\xi) = \psi^{**}_1(\underset{\sim}{i},\underset{\sim}{z},\xi) + \sum_{\nu_1=1}^{\infty} \sum_{\nu_2=0}^{\infty} \sum_{n=1}^{\infty} r_{2n}(\underset{\sim}{i},\underset{\sim}{\nu},\xi)$$

$$\cdot \psi^{**}_1(\underset{\sim}{\nu},\underset{\sim}{z},\xi) , \; i_1 \geq 1,$$

(43) $$\phi^{**}_{12}(\underset{\sim}{i},\underset{\sim}{z},\xi) = \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=1}^{\infty} \sum_{n=0}^{\infty} r_{2n+1}(\underset{\sim}{i},\underset{\sim}{\nu},\xi) \psi^{**}_2(\underset{\sim}{\nu},\underset{\sim}{z},\xi) ,$$

$$i_1 \geq 1,$$

(44) $$\phi^{**}_{21}(\underset{\sim}{i},\underset{\sim}{z},\xi) = \sum_{\nu_1=1}^{\infty} \sum_{\nu_2=0}^{\infty} \sum_{n=0}^{\infty} \tilde{r}_{2n+1}(\underset{\sim}{i},\underset{\sim}{\nu},\xi) \psi^{**}_1(\underset{\sim}{\nu},\underset{\sim}{z},\xi),$$

$$i_2 \geq 1 ,$$

(45) $$\phi^{**}_{22}(\underset{\sim}{i},\underset{\sim}{z},\xi) = \psi^{**}_2(\underset{\sim}{i},\underset{\sim}{z},\xi)$$

$$+ \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=1}^{\infty} \sum_{n=1}^{\infty} \tilde{r}_{2n}(\underset{\sim}{i},\underset{\sim}{\nu},\xi) \psi^{**}_2(\underset{\sim}{\nu},\underset{\sim}{z},\xi) ,$$

$$i_2 \geq 1 ,$$

Denoting by $\pi^*_\nu(\underset{\sim}{i},\underset{\sim}{j},\xi)$ the Laplace transform of $\pi_\nu(\underset{\sim}{i},\underset{\sim}{j},t)$ defined in (28) and (29) and

$$(46) \qquad \pi_\nu^{**}(\underset{\sim}{i},\underset{\sim}{z},\xi) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \pi_\nu^{*}(\underset{\sim}{i},\underset{\sim}{j},\xi) z_1^{j_1} z_2^{j_2} ,$$

$$|z_\nu| \leq 1, \ \nu=1,2,$$

we obtain:

Theorem 3.2

The transforms of the joint distribution of queue lengths $\xi_1(t)$ and $\xi_2(t)$ and the type of the unit served at t are given by:

(47)

$$\pi_1^{**}(\underset{\sim}{i},\underset{\sim}{z},\xi) = \phi_{11}^{**}(\underset{\sim}{i},\underset{\sim}{z},\xi)+\theta_1^{i_1}(\xi)\theta_2^{i_2}(\xi)[\xi+\lambda_1+\lambda_2-\lambda_1\theta_1(\xi)-\lambda_2\theta_2(\xi)]^{-1}$$

$$[\lambda_1\phi_{11}^{**}(1,0;\underset{\sim}{z},\xi) + \lambda_2\phi_{21}^{**}(0,1;\underset{\sim}{z},\xi)], \ i_1 \geq 1,$$

and:

(48)

$$\pi_2^{**}(\underset{\sim}{i},\underset{\sim}{z},\xi) = \phi_{12}^{**}(\underset{\sim}{i},\underset{\sim}{z},\xi) + \theta_1^{i_1}(\xi)\theta_2^{i_2}(\xi)[\xi+\lambda_1+\lambda_2-\lambda_1\theta_1(\xi)-\lambda_2\theta_2(\xi)]^{-1}$$

$$[\lambda_1\phi_{12}^{**}(1,0;\underset{\sim}{z},\xi) + \lambda_2\phi_{22}^{**}(0,1;\underset{\sim}{z},\xi)] , \ i_1 \geq 1$$

where $\phi_{r\nu}^{**}(\cdot,\cdot,\cdot)$ are given in (42) through (45).

Proof:

The result follows from the usual renewal argument given in Chapter I. For a complete proof we refer to Neuts and Yandin (1968).

5. Applications

There is a large class of application in which the priority assignment follows more naturally from the nature of the service demanded than from the urgency with which the service is needed. In many practical applications a switch of service from one class of items to another involves a set up cost or set up time. The classification of the input items according to similarity of service requirements is hence desirable. The alternating priority model was first discussed by Avi-Itzhak, Maxwell and Miller (1965). They considered the alternating priority model with zero switching. The model we considered in Chapter III is the non-zero switching case which is a generalization to zero-switch. Although the analysis of the non-zero switching model is very complicated, it is more practical. In the case of a device controlling traffic at an intersection, the zero switch rule allows one stream of vehicles access to the intersection as long as there are vehicles in this stream and a steady input of vehicles in this stream delays other streams indefinitely. A compromise rule is to allow a certain number k_1 of vehicles of one stream access to the intersection and then that stream is stopped and to allow a certain number k_2 from another stream, etc. The optimum numbers k_1 and k_2 may then depend on traffic conditions.

CHAPTER IV

A PRIORITY RULE BASED ON THE RANKING OF THE SERVICE TIMES FOR THE M|G|1 QUEUE

1. Concepts and Definitions

This chapter presents mainly the content of the article by Nair and Neuts (1969). Here we propose a priority rule based on the length of service demanded by a customer. Takács (1964) discussed a priority queue based on the rankings of the service times of the customers and obtained the asymptotic moments of the virtual waiting time assuming that a customer with a shorter service time has priority over a customer with longer service time. Here we consider a different, but related problem.

We first recall a branching process description of the M|G|1 queue suggested by Kendall (1951) and investigated by Neuts (1969). Suppose that at time $t=0$ there are $i \geq 1$ customers in the queue and that one of them is just entering service at that time. These customers form the first generation and their total service time is the lifetime of the first generation. Customers arriving during the

lifetime of the first generation, if any, make up the second generation, with its lifetime, and so on. If at the end of the first or a subsequent generation's lifetime there are no customers in the queue, then there is an idle period at the end of which a customer arrives who makes up the first generation of a busy period.

It is clear that the life time of a generation does not depend on the order in which customers have been served during it. We will study the virtual waiting time for the $M|G|1$ queue under the assumption that within each generation customers are served in the order of shortest (or longest) service times. We will call these policies the shortest processing time (SPT) and the longest processing time (LPT) disciplines, respectively, and compare them to the first-come, first-served (FCFS) discipline. Once the rearrangement is achieved within a generation, the incoming customers thereafter do not upset it; hence the question of service preemption does not arise here.

2. The Basic Imbedded Semi-Markov Process

We assume that at t=0 there are $i > 0$ customers in the queue and that the one with shortest service time enters service immediately. A sequence of random variables $T_0, T_1, \ldots$ is defined as follows: $T_0 = 0$ and T_n is the time at which all customers, if any, present at T_{n-1} complete service; if there are no customers at T_{n-1}, then T_n is the

time at which the first customer who arrives after T_{n-1} completes service. That is, T_n is the time of service completion of n-th generation, if the n-th generation is not empty. On the other hand, if the n-th generation is empty, then T_n is the time of service completion of the first customer who initiates the first busy period after time T_{n-1}.

Let $\xi(t)$ denote the number of customers in the system, at time t+0, who still require some service. Then the bivariate sequence of random variables:

(1) $$\{\xi(T_n),\ T_{n+1} - T_n,\ n \geq 0\}$$

is a Semi-Markov sequence.

We define the taboo probabilities:

(2) $${}_oQ_{ij}^{(o)}(x) = \delta_{ij}\ U(x),$$

and

$${}_oQ_{ij}^{(n)}(x) = P\Big\{T_n \leq x,\ \xi(T_n)=j, \xi(T_\nu)\neq 0, \nu=1,2,\ldots,n-1$$
$$\Big|\ \xi(T_o)=i\Big\}\ ,\ n \geq 1$$

3. The Virtual Waiting time Process

Consider an $M|G|1$ queue that has a Poisson input with parameter λ and a continuous service-time distribution function $H(\cdot)$ with finite mean α. We denote by $\underline{\eta}(t,x)$ the waiting time of a virtual customer arriving at t whose service time is $x > 0$, where the $M|G|1$ queue observes an SPT discipline, and $\bar{\eta}(t,x)$ the corresponding virtual waiting time in an $M|G|1$

queue with an LPT discipline. Let

$$(3)\quad W_i(t,x,y) = P\left\{0 \le \underline{\eta}(t,x) \le y \mid \xi(0) = i\right\},$$

$$(4)\quad \Lambda_i(t,x,y) = P\left\{0 < \underline{\eta}(t,x) \le y,\ \underline{\eta}(\tau,x) \ne 0 \right.$$

$$\left. \text{for all } \tau \in (0,t] \mid \xi(0) = i\right\}$$

Then as in (1.61) we have:

$$(5)\quad W_i(t,x,y) = \Lambda_i(t,x,y) + \int_0^t \Lambda_1(t-\tau,x,y)\, dM_1(\tau)$$

$$+ P\left\{\underline{\eta}(t,x) = 0 \mid \xi(0) = i\right\} U(y)$$

Let $W_i^*(t,x,s)$ and $\Lambda_i^*(t,x,s)$ respectively be the L.S.T. of $W_i(t,x,y)$ and $\Lambda_i(t,x,y)$ with respect to the variable y and let $W_i^{**}(\xi,x,s)$ and $\Lambda_i^{**}(\xi,x,s)$ respectively be the Laplace transforms of $W_i^*(t,x,s)$ and $\Lambda_i^*(t,x,s)$ with respect to t. Further we denote:

$$(6)\qquad \widetilde{H}(z) = \frac{H(z)}{H(x)} \quad \text{if } 0 \le z \le x,$$

$$= 0 \text{ otherwise,}$$

and $h(s)$, $\widetilde{h}(s)$, ${}_oq_{ij}^{(n)}(s)$ the L.S.T. of $H(\cdot)$, $\widetilde{H}(\cdot)$ and ${}_oQ_{ij}^{(n)}(\cdot)$ respectively, and

$$(7)\qquad {}_oq_i^{(n)}(s,z) = \sum_{j=0}^{\infty} {}_oq_{ij}^{(n)}(s)\, z^j, \quad |z| \le 1,$$

Lemma 4.1

For $R(s) > 0$ and $R(\xi) \geq 0$, the transform $\Lambda_1^{**}(\xi,x,s)$ of $\Lambda_1(t,x,y)$ is given by:

$$(8) \quad \Lambda_1^{**}(\xi,x,s) = \frac{1}{\xi-s} \sum_{n=0}^{\infty} \left[h_n^i(\xi,Z) - h_n^i(\xi,Z')\right],$$

where:

$$(9) \quad Z = h\{s+\lambda H(x)[1-\tilde{h}(s)]\},\ Z' = h\{\xi+\lambda H(x)[1-\tilde{h}(s)]\},$$

and

$$(10) \quad h_o(\xi,z) = z,\ h_n(\xi,z) = h\left[\xi+\lambda-\lambda h_{n-1}(\xi,z)\right],\ n \geq 1,$$

Proof:

We have:

(11)

$$\Lambda_1(t,x,y) = \int_0^t \int_t^{y+t} \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \sum_{\nu=0}^{\infty} d_o Q_{ij}^{(n)}(u) e^{-\lambda(v-u)} \frac{[\lambda(v-u)]^\nu}{\nu!}$$
$$(u)(v)$$

$$\sum_{k=0}^{\nu} \binom{\nu}{k} H^k(x)[1-H(x)]^{\nu-k}\, \tilde{H}^{(k)}(y-v+t) d_v H^{(j)}(v-u),$$

where $\tilde{H}(\cdot)$ is defined in (6), $H^{(m)}(\cdot)$ and $\tilde{H}^{(m)}(\cdot)$ are the m-fold convolutions of $H(\cdot)$ and $\tilde{H}(\cdot)$. The probabilistic argument to get (11) is the following: If the queue has never become empty in $(0,t]$, let the last beginning of the life of a generation occur between u and u+du and let there be j individuals in that generation. Let the end of the life time of that generation be between v and v+dv ($v > t$). In the

interval (u,v), $\nu \geq 0$ customers arrive, and they have priority over the virtual customer if and only if their service time does not exceed x. If there are k such customers, $0 \leq k \leq \nu$, then the distribution of their total service time is $\tilde{H}^{(k)}(\cdot)$. The formula (11) is obtained by using the independence properties and summing over all allowable values of n, j, ν, k, u and v.

Taking the transforms of (11):

(12)

$$\Lambda_i^*(t,x,s) = e^{st}\int_{\substack{o\\(u)}}^{t}\int_{\substack{t\\(v)}}^{\infty}\sum_{n=0}^{\infty}\sum_{j=1}^{\infty} d_o Q_{ij}^{(n)}(u)e^{-(v-u)\{s+\lambda H(x)[1-\tilde{h}(s)]\}}$$

$$\cdot\ d_v\ H^{(j)}(v-u)$$

(13)

$$\Lambda_i^{**}(\xi,x,s) = \frac{1}{\xi-s}\sum_{n=0}^{\infty}\sum_{j=1}^{\infty} {}_oq_{ij}^{(n)}(\xi)\ \{h^j[s+\lambda H(x)(1-\tilde{h}(s))]$$

$$- h^j[\xi+\lambda H(x)(1-\tilde{h}(s))]\}$$

$$= \frac{1}{\xi-s}\sum_{n=0}^{\infty}\{{}_oq_i^{(n)}(\xi,Z) - {}_oq_i^{(n)}(\xi,Z')\},$$

where Z and Z′ are defined in (9).

Now the lemma follows from lemma 1 in Appendix C.

Theorem 4.1

For $R(s) > 0$ and $R(\xi) \geq 0$ the transform $W_i^{**}(\xi,x,s)$ of the distribution function of the virtual waiting time $\eta(t,x)$ is given by:

$$(14)\qquad W_i^{**}(\xi,x,s) = \frac{1}{(\xi-s)} \sum_{n=0}^{\infty} \left[h_n^i(\xi,Z) - h_n^i(\xi,Z')\right]$$

$$+ \frac{\gamma^i(\xi)}{\xi+\lambda-\lambda\gamma(\xi)}\left\{1 + \frac{\lambda}{(\xi-s)} \sum_{n=0}^{\infty}\left[h_n(\xi,Z)-h_n(\xi,Z')\right]\right\}$$

where Z and Z' are given by (9) and $h_n(\cdot,\cdot)$ by (10).

Proof:

Taking the transform of (5) we get, as in Theorem 1.2, that:

$$(15)\quad W_i^{**}(\xi,x,s) = \Lambda_i^{**}(\xi,x,s) + \frac{\gamma^i(\xi)}{\xi+\lambda-\lambda\gamma(\xi)}\left[1+\lambda\Lambda_1^{**}(\xi,x,s)\right]$$

The theorem now follows from lemma 4.1.

Limiting Behavior of Virtual Waiting time Process

Let $W(x,y) = \lim_{t\to\infty} W_i(t,x,y)$. The existence of this limiting distribution can be demonstrated as in Theorem 1.3.

Theorem 4.2

The L.S.T. $\omega(x,s)$ of the limiting distribution $W(x,y)$ of the virtual waiting time $\eta(t,x)$ is given by:

$$(16)\quad \omega(x,s) = (1-\lambda\alpha)\left\{1 - \frac{\lambda}{s}\sum_{n=0}^{\infty}\left[h_n(0,Z)-h_n(0,\hat{Z})\right]\right\}, \text{ if } 1-\lambda\alpha > 0,$$

$$= 0 \text{ otherwise,}$$

where $h_n(\cdot,\cdot)$ are given by (10) and

$$(17)\ Z = h\left\{s+\lambda H(x)\left[1-\tilde{h}(s)\right]\right\},\ \tilde{Z} = h\left\{\lambda H(x)\left[1-\tilde{h}(s)\right]\right\}$$

<u>Proof</u>:

Similar to the proof of Theorem 1.3. As in (1.65) we obtain:

$$(18)\qquad \omega(x,s) = (1-\lambda\alpha)\ [1 + \lambda\ \Lambda_1^{**}(0,x,s)] \text{ if } 1-\lambda\ \alpha > 0$$

$$= 0 \text{ otherwise,}$$

Substitution of lemma 4.1 in (18) proves the theorem.

Taking the limit as $s \to 0+$ in (16) we observe that $\omega(x,0+) = 1$.

The Moments of the Limiting Distribution

We use the following notation:

$$\beta = \int_0^\infty u^2\, dH(u)\ ,$$

$$\gamma = \int_0^\infty u^3\, dH(u),$$

$$\alpha_x = \int_0^x u\, dH(u)\ ,$$

$$\beta_x = \int_0^x u^2\, dH(u)\ ,$$

$$\gamma_x = \int_0^x u^3\, dH(u)\ ,$$

and

$$(19)\qquad \psi_n(x,s) = h_n(0,Z) - h_n(0,\tilde{Z})\ ,$$

where Z and $\tilde{Z}$ are given in (17).

In terms of the functions ψ_n formula (16) yields:

$$(20) \qquad \omega(x,s) = (1-\lambda\alpha)\left\{1 - \lambda \sum_{n=0}^{\infty} \frac{\psi_n(x,s)}{s}\right\},$$

By lemma 3 in Appendix C the series $\sum_{n=1}^{\infty} \psi_n(x,s)$ is dominated by a convergent series if $1-\lambda\alpha > 0$. Hence by Lebesgue dominated convergence theorem, term by term differentiation gives:

$$(21) \quad -\frac{\partial}{\partial s}\,\omega(x,s)\Big]_{s=0} = \lambda(1-\lambda\alpha) \sum_{n=0}^{\infty} \frac{s\psi_n'(x,s) - \psi_n(x,s)}{s^2}\Big]_{s=0}$$

Applying l'Hopital's rule twice we get:

$$(22) \qquad -\frac{\partial}{\partial s}\,\omega(x,s)\Big]_{s=0} = \frac{\lambda(1-\lambda\alpha)}{2} \sum_{n=0}^{\infty} \psi_n''(x,0)$$

where the number of primes denotes the number of derivatives taken in succession with respect to s. Similarly:

$$(23) \qquad \frac{\partial^2\omega(x,s)}{\partial s^2}\Big]_{s=0} = -\frac{\lambda(1-\lambda\alpha)}{3} \sum_{n=0}^{\infty} \psi_n'''(x,0)$$

From (19) we have for $n \geq 0$,

(24)

$$\psi_n''(x,0) = h_n''(0,1)\left[\left(\frac{\partial Z}{\partial s}\right)^2 - \left(\frac{\partial \tilde{Z}}{\partial s}\right)^2\right]_{s=0} + h_n'(0,1)\left[\frac{\partial^2 Z}{\partial s^2} - \frac{\partial^2 \tilde{Z}}{\partial s^2}\right]_{s=0}$$

$$(25) \qquad \psi_n'''(x,0) = h_n'''(0,1)\left[\left(\frac{\partial Z}{\partial s}\right)^3 - \left(\frac{\partial \tilde{Z}}{\partial s}\right)^3\right]_{s=0}$$

$$+ 3h_n''(0,1)\left[\frac{\partial Z}{\partial s}\frac{\partial^2 Z}{\partial s^2} - \frac{\partial \tilde{Z}}{\partial s}\frac{\partial^2 \tilde{Z}}{\partial s^2}\right]_{s=0}$$

$$+ h_n'(0,1)\left[\frac{\partial^3 Z}{\partial s^3} - \frac{\partial^3 \tilde{Z}}{\partial s^3}\right]_{s=0}$$

Further it follows from (17) that:

$$\left.\frac{\partial Z}{\partial s}\right]_{s=0} = -\alpha\,(1 + \lambda\,\alpha_x)\ ,$$

$$\left.\frac{\partial \tilde{Z}}{\partial s}\right]_{s=0} = -\lambda\,\alpha\,\alpha_x,$$

$$\left.\frac{\partial^2 Z}{\partial s^2}\right]_{s=0} = \beta(1 + \lambda\,\alpha_x)^2 + \lambda\,\alpha\,\beta_x\ ,$$

$$\left.\frac{\partial^2 \tilde{Z}}{\partial s^2}\right]_{s=0} = \lambda^2\,\beta\,\alpha_x^2 + \lambda\,\alpha\,\beta_x\ ,$$

$$\left.\frac{\partial^3 Z}{\partial s^3}\right]_{s=0} = -\gamma(1+\lambda\alpha_x)^3 - 3\lambda\beta\beta_x(1+\lambda\alpha_x) - \lambda\alpha\,\gamma_x\ ,$$

$$\left.\frac{\partial^3 \tilde{Z}}{\partial s^3}\right]_{s=0} = -\gamma(\lambda\alpha_x)^3 - 3\lambda^2\beta\,\beta_x\,\alpha_x - \lambda\,\alpha\,\gamma_x\ ,$$

Substituting these calculations in (24) and (25) and summing over n with the help of lemma 2 in Appendic C we obtain:

$$(26) \qquad \sum_{n=0}^{\infty}\psi_n''(x,0) = \frac{\beta(1 + 2\lambda\,\alpha_x)}{(1-\lambda\alpha)(1-\lambda^2\alpha^2)}$$

and

$$(27) \qquad \sum_{n=0}^{\infty} \psi_n'''(x,0) = \frac{-1}{(1-\lambda\alpha)(1-\lambda^2\alpha^2)} \Big\{ 3\lambda\beta\beta_x + (1-\lambda^3\alpha^3)^{-1}$$

$$\cdot\ (1+3\lambda\alpha_x+3\lambda^2\alpha_x^2)[\gamma(1-\lambda^2\alpha^2)+3\lambda^2\alpha\beta^2]\Big\}$$

Let $M_{\underline{\eta}}(x)$ and $V_{\underline{\eta}}(x)$ denote respectively the first and second moments of the limiting distribution of $\underline{\eta}(t,x)$. Substitution of (26) and (27) into (22) and (23) respectively leads to:

$$(28) \qquad M_{\underline{\eta}}(x) = \frac{\lambda\ \beta(1 + 2\lambda\ \alpha_x)}{2(1 - \lambda^2\ \alpha^2)}$$

and

$$(29)\ V_{\underline{\eta}}(x) = \frac{\lambda}{3(1-\lambda^2\alpha^2)} \Big\{ 3\lambda\beta\beta_x + (1-\lambda^3\alpha^3)^{-1}(1+3\lambda\alpha_x + 3\lambda^2\alpha_x^2)$$

$$\cdot\ \left[\gamma(1-\lambda^2\alpha^2)+3\lambda^2\alpha\ \beta^2\right] \Big\}$$

4. The Longest Processing Time Discipline

In the longest processing time (LPT) discipline, within each generation the customers are ordered according to their length of service times, with highest priority going to the customer with longest service time. The virtual waiting time process of the present case can be treated as in the case of SPT discipline. As we have denoted $\overline{\eta}(t,x)$ is the virtual waiting time of a customer arriving at t whose service time is $x > 0$ in the case of LPT discipline. The Laplace-Stieltjes transform of the limiting distribution of $\overline{\eta}(t,x)$ can be obtained as in (16):

$$(30) \qquad (1-\lambda\alpha)\left\{1 - \frac{\lambda}{s} \sum_{n=0}^{\infty}\left[h_n(0,\zeta) - h_n(0,\tilde{\zeta})\right]\right\}, \text{ for } 1-\lambda\alpha > 0,$$

where

$$(31) \qquad \zeta = h\{s + \lambda[1-H(x)]\,[1 - \tilde{\tilde{h}}(s)]\}\ ,$$

$$\tilde{\zeta} = h\{\lambda[1 - H(x)]\,[1 - \tilde{\tilde{h}}(s)]\}\ ,$$

$\tilde{\tilde{h}}(\cdot)$ is the Laplace-Stieltjes transform of $\tilde{\tilde{H}}(\cdot)$

and

$$(32) \qquad \tilde{\tilde{H}}(z) = \frac{H(z) - H(x)}{1 - H(x)}\ , \text{ if } z > x$$

$$= 0 \text{ otherwise}\ ,$$

The first and second moments of the limiting distribution of $\bar{\eta}(t,x)$ are obtained from (30) as:

$$(33) \qquad M_{\bar{\eta}}(x) = \frac{\lambda\beta(1 + 2\lambda\alpha_x^*)}{2(1 - \lambda^2\alpha^2)}$$

and

$$(34) \quad V_{\bar{\eta}}(x) = \frac{\lambda}{3(1-\lambda^2\alpha^2)}\left\{3\lambda\beta\beta_x^* + (1-\lambda^3\alpha^3)^{-1}(1+3\lambda\alpha_x^*+3\lambda^2\alpha_x^{*2})\right.$$
$$\left. \cdot\left[\gamma(1-\lambda^2\alpha^2) + 3\lambda^2\alpha\beta^2\right]\right\}$$

where

$$\alpha_x^* = \alpha - \alpha_x\ ,\ \beta_x^* = \beta - \beta_x$$

5. Comparison of the SPT, LPT and FCFS Disciplines

Let $\eta(t)$ be the virtual waiting time of a customer arriving at t in an $M|G|1$ queue with FCFS discipline and let M_η be the mean of the limiting distribution of $\eta(t)$. Then it is known that:

(35) $$M_{\eta} = \frac{\lambda\beta}{2(1-\lambda\alpha)} ,$$

From (28), (33) and (35) we observe an interesting relationship among $M_{\underline{\eta}}(x)$, $M_{\overline{\eta}}(x)$ and M_{η}:

(36) $$M_{\eta} = \frac{1}{2}\left[M_{\underline{\eta}}(x) + M_{\overline{\eta}}(x)\right]$$

Also,

(37) $$M_{\underline{\eta}}(x) \leq M_{\eta} \leq M_{\overline{\eta}}(x) \text{ if and only if } \alpha_x \leq \frac{\alpha}{2} ,$$

Again, $M_{\underline{\eta}}(X)$ and $M_{\overline{\eta}}(X)$ are random variables with respect to X, which has a distribution function $H(\cdot)$. If we denote by E_X the expectation with respect to the random variable X, then:

$$E_X\, M_{\underline{\eta}}(X) = \int_0^\infty M_{\underline{\eta}}(x)\, dH(x)$$

(38) $$= \frac{\lambda\beta}{2(1-\lambda^2\alpha^2)}\left[1 + 2\lambda\alpha - 2\lambda\int_0^\infty u\, H(u)\, dH(u)\right] ,$$

and

(39) $$E_X\, M_{\overline{\eta}}(X) = \frac{\lambda\beta}{2(1-\lambda^2\alpha^2)}\left[1 + 2\lambda\int_0^\infty u\, H(u)\, dH(u)\right] ,$$

In particular if $H(x) = 1 - e^{-\mu x}$, $x \geq 0$, then:

(40) $$E_X\, M_{\underline{\eta}}(X) = \frac{\rho(2+\rho)}{2\mu(1-\rho^2)} ,$$

(41) $$E_X\, M_{\overline{\eta}}(X) = \frac{\rho(2+3\rho)}{2\mu(1-\rho^2)} ,$$

and

(42) $$M_{\eta} = \frac{\rho}{\mu(1-\rho)} ,$$

where ρ is the traffic intensity $\frac{\lambda}{\mu}$. Hence in the case of an $M|M|1$ queue:

(43) $$E_x\, M_{\underline{\eta}}(X) \leq M_\eta \leq E_x\, M_{\overline{\eta}}(X) \ ,$$

Further examination reveals that:

(44) $$M_{\underline{\eta}}(x) = \left(\frac{1 + 2\lambda\, \alpha_x}{1 + \lambda\, \alpha}\right) M_\eta \ ,$$

That is, the steady state expected virtual waiting time under SPT rule is obtained from the steady state expected virtual waiting time under FCFS rule by multiplying with the factor $(1 + 2\lambda\, \alpha_x) \mid (1 + \lambda\alpha)$ which increases monotonically from $1/(1 + \lambda\alpha)$ for $x = 0$ to $(1 + 2\lambda\alpha)/(1 + \lambda\alpha)$ for $x \to \infty$. The factor $(1 + 2\lambda\alpha_x)/(1 + \lambda\alpha) > 1/(1 + \lambda\alpha) > \frac{1}{2}$, since $\lambda\alpha < 1$ by the steady state condition. Hence:

$$M_{\underline{\eta}}(x) > \frac{1}{2} M_\eta \quad \text{for all } x \geq 0,$$

and $M_{\underline{\eta}}(x) \downarrow \frac{1}{2} M_\eta$ as $\lambda\alpha \uparrow 1$ and **for small x**. Again, $(1+2\lambda\alpha_x)/(1+\lambda\alpha) \leq (\frac{1+2\lambda\alpha}{1+\lambda\alpha}) = 2 - 1/(1+\lambda\alpha) < \frac{3}{2}$ which implies that:

$$M_{\underline{\eta}}(x) < \frac{3}{2} M_\eta \ \text{for all } x \geq 0,$$

and $M_{\underline{\eta}}(x) \uparrow \frac{3}{2} M_\eta$ as $\lambda\alpha \uparrow 1$ and for large x Thus:

(45) $$\frac{1}{2} M_\eta \leq M_{\underline{\eta}}(x) \leq \frac{3}{2} M_\eta$$

Similarly we observe:

(46) $$M_{\overline{\eta}}(x) = \left(\frac{1 + 2\,\lambda\,\alpha_x^*}{1 + \lambda\,\alpha}\right) M_\eta \quad ,$$

If we draw the graphs of $y = M_{\underline{\eta}}(x)$ and $y = M_{\overline{\eta}}(x)$, then it is easily seen that they are symmetrically situated on either side of the line $y = M_\eta$. Hence whenever $M_{\underline{\eta}}(x)$ satisfies

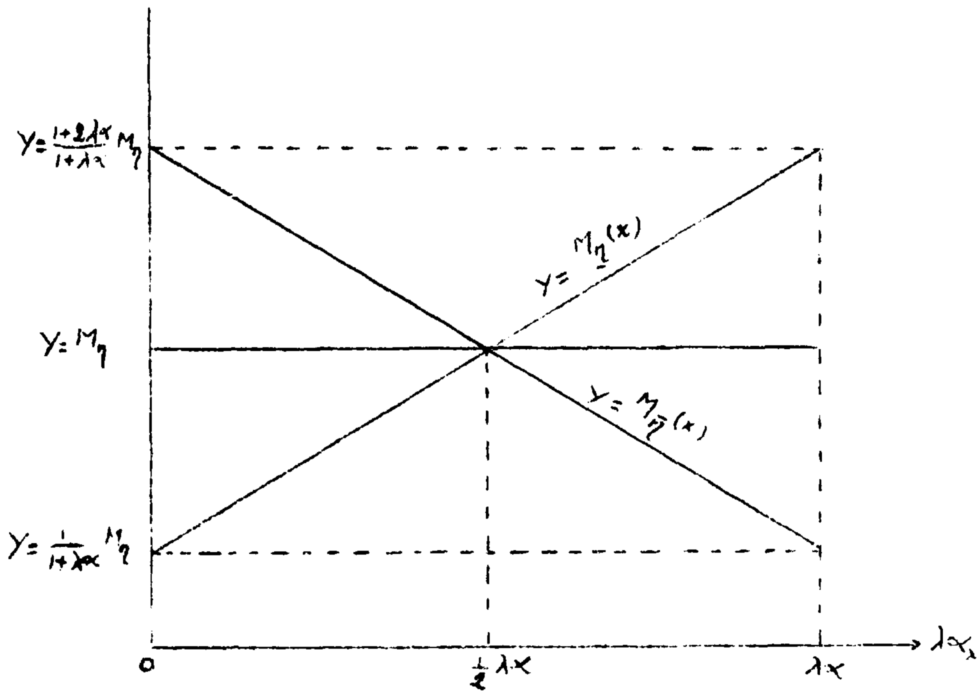

Figure 3.

The Comparison Graph

the inequality (45) $M_{\overline{\eta}}(x)$ also satisfies the same inequality but realizes in the reverse direction. They are concurrent with M_η when $\alpha_x = \frac{\alpha}{2}$. This is graphically shown in Figure 3.

6. A Renewal Argument

Equation (11) can be written as:

(47)

$$\Lambda_i(t,x,y) = \int_o^t \sum_{j=1}^{\infty} {}_oR_{ij}(du) \int_{t-u}^{t-u+y} H^{(j)}(dz) \sum_{\nu=0}^{\infty} e^{-\lambda z} \frac{(\lambda z)^{\nu}}{\nu!}$$

$$\cdot \sum_{k=0}^{\nu} \binom{\nu}{k} H^k(x)[1-H(x)]^{\nu-k} \tilde{H}^{(k)}(t+y-u-z)$$

which is obtained by replacing v-u by z in (11) and defining $\sum_{n=0}^{\infty} {}_oQ_{ij}^{(n)}(t) = {}_oR_{ij}(t)$. ${}_oR_{ij}(t)$ is the expected number of times state j is entered without visiting the state zero in $[0,t]$, starting at state i. Defining:

(48)

$$F_j(t-u,x,y) = \int_{t-u}^{t-u+y} H^{(j)}(dz) \sum_{\nu=0}^{\infty} e^{-\lambda z} \frac{(\lambda z)^{\nu}}{\nu!} \sum_{k=0}^{\nu} \binom{\nu}{k} H^k(x)$$

$$\cdot \left[1 - H(x)\right]^{\nu-k} \tilde{H}^{(k)}(t+y-u-z) ,$$

we rewrite (47) as:

$$(49) \qquad \Lambda_i(t,x,y) = \int_o^t \sum_{j=1}^{\infty} {}_oR_{ij}(du) \, F_j(t-u,x,y) ,$$

(49) together with (5) gives:

$$(50) \qquad W_i(t,x,y) = \int_o^t \sum_{j=1}^{\infty} {}_oR_{ij}(du) F_j(t-u,x,y)$$

$$+ \int_o^t \Lambda_1(t-\tau,x,y) \, dM_1(\tau)$$

$$+ P\{\eta(t,x) = 0 \mid \xi(0) = i\} \, U(y)$$

By Smith's Key Renewal Theorem (Theorem 4, Appendix D) and lemma 1.8:

$$(51) \qquad W(x,y) = \lim_{t \to \infty} W_1(t,x,y)$$

$$= \int_0^\infty \sum_{j=1}^\infty \frac{1}{{}_0\mu_j} F_j(\tau,x,y)d\tau + \frac{1}{\mu}\int_0^\infty \Lambda_1(\tau,x,y)d\tau$$

$$+ (1-\lambda\alpha)\, U(y) \text{ , if } 1-\lambda\, \alpha > 0 \text{ ,}$$

$$= 0 \text{ otherwise}$$

where ${}_0\mu_j$ is the mean recurrence time of state j without visiting state 0 and μ is the mean renewal time of the general renewal process formed by the beginning of busy periods.

From (1.52) and (2.86):

$${}_0\mu_j^{-1} = 0 \text{ and } \mu^{-1} = \lambda(1-\lambda\alpha)$$

Hence further simplification of (51) gives:

(52)

$$W(x,y) = (1-\lambda\alpha)\left\{U(y)+\lambda \sum_{j=1}^\infty {}_0R_{1j}(+\infty)\int_0^\infty F_j(\tau,x,y)\, d\tau\right\} \text{ ,}$$

$$\text{if } 1-\lambda\alpha > 0,$$

$$= 0 \text{ otherwise,}$$

7. An Exact Comparison of the Waiting times Under Three Priority Rules

A number of comparisons between SPT, LPT and FCFS rules were carried out in the earlier sections in regards to expected waiting times in the equilibrium state. Many questions which involve more than expected values may be asked however and in order to answer them an exact comparison of the waiting times as random variables needs to be made.

We may "visualize" the definition of the three random variables $\underline{\eta}(t,x)$, $\overline{\eta}(t,x)$, $\eta(t)$ on a common probability space as follows. Imagine that a customer joining the queue at time t consists of three identical parts 1, 2, 3 all requiring a processing time $x \geq 0$. Part 1 waits in front of a server operating under the SPT rule, part 2 in front of a server operating under the LPT rule and finally part 3 waits in front of a unit governed by the FCFS rule. Then $\underline{\eta}(t,x)$, $\overline{\eta}(t,x)$ and $\eta(t)$ are the waiting times of parts 1, 2 and 3 respectively.

An Auxiliary Calculation

Consider the time points t and $t+t'$, $t > 0$, $t' > 0$. The probability that during the interval t, j_1 customers arrive whose service time is less than x, j_2 whose service time is greater than x and that during $(t,t+t')$ j_3 and j_4 arrive with service times respectively less and greater than x is given by:

$$(53)\qquad e^{-\lambda t-\lambda t'}\frac{[\lambda tH(x)]^{j_1}}{j_1!}\frac{[\lambda t'H(x)]^{j_3}}{j_3!}\frac{\{\lambda t[1-H(x)]\}^{j_2}}{j_2!}\cdot\frac{\{\lambda t'[1-H(x)]\}^{j_4}}{j_4!}$$

We assume that x is a point of continuity of $H(\cdot)$ so that the probability that one or more customers have service time exactly equal to x is zero.

Let U_1' and U_2' be the total service time of all customers in $(0,t)$ with service time respectively less and greater than x. Similarly U_3' and U_4' are the corresponding quantities for the customers arriving in $(t,t+t')$.

We define $W(t,t'\,;x_1,x_2,x_3,x_4)$ as the probability that for given $t > 0$ and $t' > 0$, the random variables U_1', U_2', U_3', U_4' satisfy:

$$U_1' \le x_1,\ U_2' \le x_2,\ U_3' \le x_3,\ U_4' \le x_4\ .$$

It follows from (53) that:

$$(54)\qquad W(t,t';x_1,x_2,x_3,x_4) = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}\sum_{j_4=0}^{\infty} e^{-\lambda t-\lambda t'}$$

$$\cdot\frac{[\lambda t\,H(x)]^{j_1}}{j_1!}\frac{[\lambda t'H(x)]^{j_3}}{j_3!}\frac{\{\lambda t[1-H(x)]\}^{j_2}}{j_2!}$$

$$\cdot\frac{[\lambda t'[1-H(x)]]^{j_4}}{j_4!}\tilde{H}^{(j_1)}(x_1)\hat{\tilde{H}}^{(j_3)}(x_3)\hat{\hat{H}}^{(j_2)}(x_2)\tilde{\hat{H}}^{(j_4)}(x_4)$$

where $\tilde{H}$ and $\tilde{\tilde{H}}$ are defined in (6) and (32) respectively. Upon taking Laplace-Stieltjes transforms:

$$(55)\ W^*(t,t';s_1,s_2,s_3,s_4) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1x_1-s_2x_2-s_3x_3-s_4x_4}$$

$$\cdot\ d_{x_1,\ldots,x_4} W(t,t';x_1,x_2,x_3,x_4)$$

we obtain:

$$(56)\ W^*(t,t';s_1,s_2,s_3,s_4) = \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty \sum_{j_3=0}^\infty \sum_{j_4=0}^\infty e^{-\lambda t-\lambda t'}$$

$$\frac{[\lambda tH(x)]^{j_1}}{j_1!} \frac{[\lambda t'H(x)]^{j_3}}{j_3!} \frac{\{\lambda t[1-H(x)]\}^{j_2}}{j_2!} \frac{\{\lambda t'[1-H(x)]\}^{j_4}}{j_4!}$$

$$\tilde{h}^{j_1}(s_1)\tilde{h}^{j_3}(s_3)\ \tilde{\tilde{h}}^{j_2}(s_2)\tilde{\tilde{h}}^{j_4}(s_4)$$

$$= \exp\Big\{-\lambda t-\lambda t'+\lambda tH(x)\tilde{h}(s_1)+\lambda t[1-H(x)]\tilde{\tilde{h}}(s_2)$$

$$+\ \lambda t'H(x)\tilde{h}(s_3)+\lambda t'[1-H(x)]\tilde{\tilde{h}}(s_4)\Big\}$$

We now return to an $M|G|1$ queue, which we consider at time t. We define the following five random variables. U_0 is the length of time beyond t until the generation of customers in service at time t completes its service. U_1 and U_2 are respectively the total service times of the customers with processing times less than and greater than x who have joined the queue since the beginning of the service time of the current generation and before t. U_3 and U_4 are respectively the total

service times of the customers with processing times less than and greater than x, who join the queue during the time interval $(t,t+U_0)$.

If at time t the server is idle all the five variables are zero.

We shall express the joint distribution of the waiting times $\underline{\eta}(t,x)$, $\overline{\eta}(t,x)$ and $\eta(t)$ in terms of the joint distribution of the random variables U_j, $j=0,\ldots,4$.

The Joint Distribution of U_j, $j=0,\ldots,4$

Let ${}_0R_i(t,x_0,x_1,x_2,x_3,x_4)$ be the probability that in $(0,t)$ the queue has never become empty and that the variable U_j, $j=0,\ldots,4$ associated with the time point t satisfy $U_j \le x_j$, $j=0,\ldots,4$, given that at t=0 there were $i \ge 1$ customers in the queue, one of who was beginning his service at that time. Then:

$$(57) \qquad {}_0R_i(t,x_0,x_1,x_2,x_3,x_4) = \sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} \int_{0\,(t')}^{x_0} \int_{0\,(\tau)}^{t} d\,{}_0Q_{i\nu}^{(n)}(\tau) \cdot dH^{(\nu)}(t+t'-\tau) \cdot W(t-\tau,t';x_1,x_2,x_3,x_4)$$

The probabilistic argument for this is the following: At some time τ prior to t, the generation in service at t enters service. There are some number $\nu \ge 1$ customers in it, so that the duration of the total service time of these ν customers

has as its distribution the ν-fold convolution $H^{(\nu)}(\cdot)$ of $H(\cdot)$. If $U_o \le x_o$ must hold, then the total service time of these ν customers cannot exceed $t+x_o$. The other requirements $U_1 \le x_1$, $U_2 \le x_2$, $U_3 \le x_3$, $U_4 \le x_4$ account for the factor $W(t-\tau; t; x_1, x_2, x_3, x_4)$. The probabilities ${}_oQ_{i\nu}^{(n)}(\cdot)$ are defined by (2).

Taking L.S.T. of (57):

(58)
$$ {}_oR_i^{**}(\xi, s_o, s_1, s_2, s_3, s_4) = \int_0^\infty e^{-\xi t} dt \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-s_o x_o - s_1 x_1 - s_2 x_2 - s_3 x_3 - s_4 x_4} d_{x_o, x_1, x_2, x_3, x_4}\, {}_oR_i(t, x_o, x_1, x_2, x_3, x_4) $$

and recalling (56), we obtain:

(59)
$$ {}_oR_i^{**}(\xi, s_o, s_1, s_2, s_3, s_4) = \sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} {}_oq_{i\nu}^{(n)}(\xi) \int_{0\,(t_1)}^{\omega} \int_{0\,(x_o)}^{\infty} \exp\Big\{-\xi t_1 - \lambda t_1 - \lambda x_o - s_o x_o + \lambda t_1 H(x)\tilde{h}(s_1) + \lambda t_1 [1-H(x)]\tilde{\tilde{h}}(s_2) + \lambda x_o H(x)\tilde{h}(s_3) + \lambda x_o [1-H(x)]\tilde{\tilde{h}}(s_4)\Big\} dH^{(\nu)}(t_1 + x_o) dt_1 $$

$$ = \Big\{\xi - s_o - \lambda H(x)[\tilde{h}(s_1) - \tilde{h}(s_3)] - \lambda[1-H(x)][\tilde{\tilde{h}}(s_2) - \tilde{\tilde{h}}(s_4)]\Big\}^{-1} \sum_{n=0}^{\infty} \Big\{ {}_oq_i^{(n)}\{\xi, h[\lambda + s_o - \lambda H(x)\tilde{h}(s_3) - \lambda(1-H(x))\tilde{\tilde{h}}(s_4)]\} - {}_oq_i^{(n)}\{\xi, h[\xi + \lambda - \lambda H(x)\tilde{h}(s_1) - \lambda(1-H(x))\tilde{\tilde{h}}(s_2)]\} \Big\} $$

in terms of the functions ${}_{o}q_i^{(n)}(\xi,z)$ defined in (7).

Next, let $R_i(t,x_o,x_1,x_2,x_3,x_4)$ be the probability that at time t, the random variables U_j associated with t satisfy $U_j \leq x_j$, j=0,1,2,3,4, given that at t=0 there were i customers in the queue.

The standard regeneration argument as in (1.36) leads to:

$$(60) \qquad R_i(t,x_o,x_1,x_2,x_3,x_4) = {}_{o}R_i(t,x_o,x_1,x_2,x_3,x_4) + \int_o^t {}_{o}R_1(t-u,x_o,x_1,x_2,x_3,x_4)\, dM_1(u) + P\{\xi(t)=0 \mid \xi(0)=i\}\, U(x_1,x_2,x_3,x_4)$$

where:

$$U(x_1,x_2,x_3,x_4) = 1 \text{ if } x_j \geq 0 \text{ for all } j=1,2,3,4$$
$$= 0 \text{ otherwise.}$$

Upon taking transforms in (60) we obtain as in (15):

$$(61) \qquad R_i^{**}(\xi,s_o,s_1,s_2,s_3,s_4) = {}_{o}R_i^{**}(\xi,s_o,s_1,s_2,s_3,s_4) + \gamma^i(\xi)[\xi+\lambda-\lambda\gamma(\xi)]^{-1}\{1+\lambda\ {}_{o}R_1^{**}(\xi,s_o,s_1,s_2,s_3,s_4)\}$$

When $1-\lambda\alpha > 0$, the existence of a joint limiting distribution for U_j, j=0,1,2,3,4 is guaranteed as in Theorem 1.3. Further when $1-\lambda\alpha \leq 0$, $R_i(t,x_o,x_1,x_2,x_3,x_4)$ tends to zero for all i and $x_j \geq 0$, j=0,...,4. Since the limiting distribution exists when $1-\lambda\alpha > 0$, its transform is given by:

(62) $$R^{*}(s_0,s_1,s_2,s_3,s_4) = \lim_{\xi \to 0+} \xi\, R_1^{**}(\xi,s_0,s_1,s_2,s_3,s_4)$$

$$= (1-\lambda\alpha)\left\{1+\lambda\ {}_0R_1^{**}(0+,s_0,s_1,s_2,s_3,s_4)\right\}$$

The Joint Distribution of $\underline{\eta}(t,x)$, $\eta(t)$ and $\overline{\eta}(t,x)$

The random variables $\underline{\eta}(t,x)$, $\overline{\eta}(t,x)$ and $\eta(t)$ are, for each $t > 0$, related to the random variables U_j, $j=0,1,2,3,4$ associated with the time instant t by:

(63) $$\underline{\eta}(t,x) = U_0 + U_1 + U_3$$

$$\eta(t) = U_0 + U_1 + U_2$$

$$\overline{\eta}(t,x) = U_0 + U_2 + U_4$$

That this is indeed so, we argue for $\underline{\eta}(t,x)$. The other cases are similar. Consider a virtual customer with service time x arriving at time t. He has to wait until all customers of the present generation, if any, have been served. This is a length of time U_0. Next, in the next generation, all customers with service time less than x are served ahead of him. Regardless of the actual order of service the total amount of processing time required by all customers with service time less than x is $U_1 + U_3$. U_1 is the processing time of those who preceded him and U_3 that of those who succeeded him in the arrival sequence. We have:

(64) $\underline{\eta}(t,x)\,\zeta_1 + \eta(t)\,\zeta_2 + \overline{\eta}(t,x)\,\zeta_3$

$$= (\zeta_1+\zeta_2+\zeta_3)\,U_0 + (\zeta_1+\zeta_2)\,U_1 + (\zeta_2+\zeta_3)\,U_2$$

$$+ \zeta_1\,U_3 + \zeta_3\,U_4 \;,$$

which implies that:

(65) $$S_i^{**}(\xi,\zeta_1,\zeta_2,\zeta_3) = \int_0^\infty e^{-\xi t} E\left[e^{-\underline{\eta}(t,x)\zeta_1-\eta(t)\zeta_2-\overline{\eta}(t,x)\zeta_3}\right] dt$$

$$= R_i^{**}(\xi,\zeta_1+\zeta_2+\zeta_3,\zeta_1+\zeta_2,\zeta_2+\zeta_3,\zeta_1,\zeta_3)$$

where $R_i^{**}(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)$ is given by (61). Formula (65) shows how the joint distribution of $\underline{\eta}(t,x)$, $\eta(t)$ and $\overline{\eta}(t,x)$ is related to the basic parameters of the $M|G|1$ queue.

<u>The Limiting Joint Distribution.</u> The Limiting joint distribution of the three virtual waiting times is given by its Laplace-Stielejes transform:

(66)

$$S^*(\zeta_1,\zeta_2,\zeta_3) = (1-\lambda\alpha)\left\{1+\lambda\ {}_0R_1^{**}(0,\zeta_1+\zeta_2+\zeta_3,\zeta_1+\zeta_2,\zeta_2+\zeta_3,\zeta_1,\zeta_3)\right\}$$

where ${}_0R_1^{**}(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)$ is given by (59).

Moments of the Limiting Distribution of Basic Variables U_j, $j=0,1,2,3,4$.

Let us denote:

$$\underset{\sim}{s} = (s_0,s_1,s_2,s_3,s_4),\quad \underset{\sim}{0} = (0,0,0,0,0)$$

(67) $$\theta \equiv \theta(x,\underset{\sim}{s})$$
$$= s_0 + \lambda H(x)[\tilde{h}(s_1)-\tilde{h}(s_3)]+\lambda[1-H(x)][\tilde{\tilde{h}}(s_2)-\tilde{\tilde{h}}(s_4)]$$

(68) $$Y \equiv Y(x,\underset{\sim}{s})$$
$$= h[\lambda+s_0-\lambda H(x)\tilde{h}(s_3)-\lambda(1-H(x))\tilde{\tilde{h}}(s_4)]$$

(69) $$\tilde{Y} \equiv \tilde{Y}(x,\underset{\sim}{s})$$
$$= h[\lambda-\lambda H(x)\tilde{h}(s_1)-\lambda(1-H(x))\tilde{\tilde{h}}(s_2)]$$

$$\psi_n \equiv \psi_n(x,\underset{\sim}{s})$$
$$= h_n(0,Y) - h_n(0,\tilde{Y}), \ n \geq 0 ,$$

where the functions $h_n(\cdot,\cdot)$, $n \geq 0$ are defined in (10). It follows from (62) that:

(70) $$R^*(\underset{\sim}{s}) = (1-\lambda\alpha)\left\{1 - \frac{\lambda}{\theta} \sum_{n=0}^{\infty}\left[{}_0q_1^{(n)}(0,Y)-{}_0q_1^{(n)}(0,\tilde{Y})\right]\right\}$$
$$= (1-\lambda\alpha)\left\{1 - \lambda \sum_{n=0}^{\infty} \frac{\psi_n(x,\underset{\sim}{s})}{\theta}\right\}$$

(similar to equation (20))

Let $E_\infty U$ denote the expected value of the limiting distribution of the variable U, and $\underset{\sim}{U}$ the column vector:

(71) $$\underset{\sim}{U} = \begin{matrix} U_0 \\ U_1 \\ U_2 \\ U_3 \\ U_4 \end{matrix}$$

Since the calculations are lengthy we state the results only. The matrices of first and second moments are respectively given by:

$$(72)\qquad E_\infty \underset{\sim}{U} = \left(\frac{\partial R^*(\underset{\sim}{s})}{\partial s_i} \right]_{\underset{\sim}{s}=\underset{\sim}{0}} \Bigg)$$

$$= \begin{bmatrix} a \\ \lambda\, \alpha_x\, a \\ \lambda\, \alpha_x^*\, a \\ \lambda\, \alpha_x\, a \\ \lambda\, \alpha_x^*\, a \end{bmatrix}$$

and

$$(73)\qquad E_\infty (\underset{\sim}{U}\,\underset{\sim}{U}') = \left(\frac{\partial^2 R^*(\underset{\sim}{s})}{\partial s_i \partial s_j} \right]_{\underset{\sim}{s}=\underset{\sim}{0}} \Bigg)$$

$$= \begin{bmatrix} b & & & & \\ \frac{\lambda}{2}\alpha_x b & \lambda^2\alpha_x^2 b + \lambda\beta_x a & & & \\ \frac{\lambda}{2}\alpha_x^* b & \lambda^2\,\alpha_x\alpha_x^*\, b & \lambda^2\alpha_x^{*2}\, b + \lambda\beta_x^* a & & \\ \lambda\,\alpha_x\, b & \frac{\lambda^2}{2}\alpha_x^2\, b & \frac{\lambda^2}{2}\alpha_x\alpha_x^*\, b & \lambda^2\alpha_x^2 b + \lambda\beta_x a & \\ \lambda\,\alpha_x^*\, b & \frac{\lambda^2}{2}\alpha_x\alpha_x^*\, b & \frac{\lambda^2}{2}\alpha_x^{*2}\, b & \lambda^2\,\alpha_x\,\alpha_x^*\, b & \lambda^2\alpha_x^{*2}\, b + \lambda\beta_x^* a \end{bmatrix}$$

where α_x, β_x, α_x^*, β_x^* are defined in section 3 and section 4 and:

$$a = E_\infty U_0 = \lambda\beta/2(1-\lambda^2\alpha^2)\ ,$$

$$b = E_\infty U_0^2 = \frac{\lambda}{2}\,(1-\lambda^3\alpha^3)^{-1}[\gamma + 3\lambda^2\alpha\beta^2(1-\lambda^2\alpha^2)^{-1}]\ ,$$

That is, a is the steady state expected residual life length of the generation serving at time t and $b-a^2$ its variance.

Moments of the Limiting Distribution of $\underline{\eta}(t,x)$, $\eta(t)$ and $\bar{\eta}(t,x)$

Let us denote:

$$\Lambda = \begin{bmatrix} \underline{\eta}(t,x) \\ \eta(t) \\ \bar{\eta}(t,x) \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

so that from (63):

$$\Lambda = A\,\underset{\sim}{U}$$

(74) $$E_\infty \Lambda = A(E_\infty \underset{\sim}{U})$$

(75) $$E_\infty(\Lambda\,\Lambda') = A(E_\infty \underset{\sim}{U}\,\underset{\sim}{U}')\,A'$$

$E_\infty \underset{\sim}{U}$ and $E_\infty \underset{\sim}{U}\,\underset{\sim}{U}'$ are given by (72) and (73), and substituting these values in (74) and (75) and simplifying we obtain:

(76) $$E_\infty \Lambda = \begin{bmatrix} (1+2\lambda\alpha_x)\,a \\ (1+\lambda\iota)\,a \\ (1+2\lambda\alpha_x^*)\,a \end{bmatrix}$$

$$(77)\qquad E_\infty(\Lambda\Lambda') = \begin{bmatrix} b(1+3\lambda\alpha_x+3\lambda^2\alpha_x^2) + 2\lambda\beta_x\, a & & \\ b(1+\frac{\lambda}{2}\alpha+\frac{3}{2}\lambda\alpha_x + \frac{3}{2}\lambda^2\alpha\alpha_x)+\lambda\beta_x a & b(1+\lambda\alpha+\lambda^2\alpha^2) + \lambda\beta\, a & \\ b(1+\frac{3}{2}\lambda\alpha+3\lambda^2\alpha_x\alpha_x^*) & b(1+\frac{\lambda}{2}\alpha+\frac{3}{2}\lambda\alpha_x^* + \frac{3}{2}\lambda^2\alpha\alpha_x^*)+\lambda\beta_x^*\, a & b(1+3\lambda\alpha_x^*+3\lambda^2\alpha_x^{*2}) + 2\lambda\beta_x^*\, a \end{bmatrix}$$

The Limiting Probability

$P\{\eta(t,x) < \eta(t)\}$ as $t \to \infty$

Let $\Lambda_1(t,x_2,x_3)$ be the joint distribution of U_2 and U_3 given t and ${}_0\Lambda_1(t,x_2,x_3)$ be the probability that in $(0,t)$ the queue has never become empty and that the variables U_2 and U_3 associated with the time point t satisfy $U_2 \leq x_2$, $U_3 \leq x_3$, given that at $t=0$ there were $i \geq 1$ customers in the queue, one of who was beginning his service at that time. Then the renewal argument as in (5) leads to:

$$(78)\qquad \Lambda_1(t,x_2,x_3) = {}_0\Lambda_1(t,x_2,x_3) + \int_0^t {}_0\Lambda_1(t-u,x_2,x_3)\,dM_1(u)$$
$$+ P\{\xi(t)=0 \mid \xi(0)=i\}\, U(x_2,x_3)$$

where:

$$U(x_2,x_3) = 1 \text{ if } x_2 \geq 0 \text{ and } x_3 \geq 0 \;,$$
$$= 0 \text{ otherwise}$$

Further let $\emptyset(t,t';x_2,x_3)$ be the joint distribution of U_2 and U_3 given t and t', then as in (54):

$$(79)\ \emptyset(t,t';x_2,x_3) = \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} e^{-\lambda t(1-H(x))-\lambda t' H(x)}$$

$$\cdot \frac{[\lambda t[1-H(x)]]^{j_2}}{j_2!} \frac{[\lambda t' H(x)]^{j_3}}{j_3!} \tilde{\tilde{H}}^{(j_2)}(x_2)\tilde{H}^{(j_3)}(x_3)$$

Similar to (57) we have:

$$(80)\ {}_o\Lambda_i(t,x_2,x_3) = \sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} \int_{\substack{o\\(t')}}^{\infty} \int_{\substack{o\\(\tau)}}^{t} d\,{}_oQ_{i\nu}^{(n)}(\tau)dH^{(\nu)}(t+t'-\tau)$$

$$\cdot\ \emptyset(t-\tau,t';x_2,x_3)$$

$$= \sum_{\nu=1}^{\infty} \int_o^t d\ {}_oR_{i\nu}(\tau)\ F_\nu(t-\tau,x_2,x_3)$$

where ${}_oR_{i\nu}(\cdot)$ is defined in section 6 and:

$$(81)\ F_\nu(t-\tau,x_2,x_3) = \int_{\substack{o\\(t')}}^{\infty} dH^{(\nu)}(t+t'-\tau)\ \emptyset(t-\tau,t';x_2,x_3)$$

Substituting (80) in (78) and applying Smith's Key Renewal Theorem (Theorem 4, Appendix D) we get the limiting distribution $\Lambda(x_2,x_3)$ of $\Lambda_i(t,x_2,x_3)$ as $t\to\infty$, as in (52):

(82)

$$\Lambda(x_2,x_3)=(1-\lambda\alpha)\left\{U(x_2,x_3) + \lambda \sum_{j=1}^{\infty} {}_oR_{1j}(+\infty)\int_o^{\infty} F_j(\tau,x_2,x_3)d\tau\right\},$$

if $1 - \lambda\alpha > 0$

$= 0$ otherwise

From (63) it follows that:

$$(83)\qquad \lim_{t\to\infty} P\{\eta(t,x) < \eta(t)\} = \lim_{t\to\infty} P\{U_3 < U_2\}$$

$$= \int_0^\infty \int_0^{x_2} d\,\Lambda(x_2,x_3)$$

$$= (1-\lambda\alpha)\Big\{1+\lambda \sum_{j=1}^{\infty} {}_0R_{1j}(+\infty) \int_{0\,(x_2)}^{\infty} \int_{0\,(x_3)}^{x_2} \int_{0\,(\tau)}^{\infty} d_{x_2x_3}F_j(\tau,x_2,x_3)d\tau\Big\}$$

8. Applications

The main objectives of a priority decision are to reduce the response time, to acknowledge customer importance and urgency of request and to serve in fair order and to limit the length of wait. For the best average performance the shortest service-time-next rule may be just right. But under this rule a steady stream of shortest requests may delay a longest request indefinitely. The rule proposed in this chapter is a compromise to this, since within each generation the service request of a customer with long service time is fulfilled. Our model, of course, assumes that the service times of the customers can be ordered before hand.

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APPENDIX A

A THEOREM ON SUMMATION OF SERIES

Theorem

For a given positive integer k, the sum of the infinite series:

$$(1)\quad \sum_{n=0}^{\infty} \frac{x^n}{n!} y^{[\frac{n+\nu}{k}]k} = \frac{1}{k} \sum_{m=0}^{k-1} \frac{\omega_m(\omega_m y)^{\nu}\,(y^k-1)}{(\omega_m y-1)\, y^{k-1}} e^{\omega_m xy}$$

for all x, for all $y \neq 1$, and for all integral values of $\nu \geq 0$, where $[\frac{n}{k}]$ is the greatest integer not exceeding $\frac{n}{k}$, and $1 = \omega_0, \omega_1, \ldots, \omega_{k-1}$ are the k-th roots of unity.

Proof:

Let us denote:

$$(2)\qquad f(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\, y^{[\frac{n+\nu}{k}]k}$$

and

$$(3)\qquad \hat{f}(s,y) = \int_0^{\infty} e^{-sx} f(x,y)\, dx$$

Then:

$$(4)\qquad \hat{f}(s,y) = \sum_{n=0}^{\infty} \frac{y^{[\frac{n+\nu}{k}]k}}{s^{n+1}}$$

Suppose that $0 \leq \nu \leq k-1$, then (4) can be written as:

$$\hat{f}(s,y) = \sum_{n=0}^{k-\nu-1} \frac{1}{s^{n+1}} + y^k \sum_{n=k-\nu}^{2k-\nu-1} \frac{1}{s^{n+1}} + y^{2k} \sum_{n=2k-\nu}^{3k-\nu-1} \frac{1}{s^{n+1}} + \dots$$

$$= \sum_{n=0}^{k-\nu-1} \frac{1}{s^{n+1}} + \frac{y^k}{s^{k-\nu}} \left(\sum_{n=0}^{k-1} \frac{1}{s^{n+1}} \right) \left[1 + \left(\frac{y}{s}\right)^k + \left(\frac{y}{s}\right)^{2k} + \dots \right]$$

$$= \frac{s^{k-\nu}-1}{s^{k-\nu}(s-1)} + \frac{y^k(s^k-1)}{s^{k-\nu}(s-1)(s^k-y^k)} \quad , \text{ for } |y| < |s| ,$$

$$(5) \qquad = \frac{s^k + s^\nu(y^k-1) - y^k}{(s-1)\,(s^k-y^k)}$$

Next we consider $r\,k \leq \nu \leq (r+1)\,k-1$, $r=0,1,2,\dots$ In this case:

$$\hat{f}(s,y) = \sum_{n=0}^{\infty} \frac{y^{[\frac{n+\nu}{k}]k}}{s^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{y^{[r+\frac{(n+\nu-rk)}{k}]k}}{s^{n+1}}$$

$$(6) \qquad = y^{rk} \sum_{n=0}^{\infty} \frac{y^{[\frac{n+r'}{k}]k}}{s^{n+1}}$$

where $r' = \nu - rk$ and $0 \leq r' \leq k-1$

Hence substitution of (5) in (6) yields:

$$\hat{f}(s,y) = \frac{y^{rk}[s^k + s^{r'}(y^k-1) - y^k]}{(s-1)\,(s^k - y^k)}$$

$$(7) \qquad = \frac{y^{rk}[s^k + s^{\nu-rk}(y^k - 1) - y^k]}{(s - 1)\,(s^k - y^k)} \quad , \text{ for } |y| < |s|$$

To find the inverse transform we use Bateman (1954), Tables of Integral Transforms (p232).

That is, if $\hat{f}(s) = \frac{Q(s)}{P(s)}$,

where $P(s) = (s-\alpha_1)\ldots(s-\alpha_n)$, $\alpha_i \neq \alpha_r$ for $i \neq r$ and $Q(s)$ is a polynomial of degree $\leq n-1$, then the inverse transform of $\hat{f}(s)$ is given by:

$$f(x) = \sum_{m=1}^{n} \frac{Q(\alpha_m)}{P_m(\alpha_m)} e^{\alpha_m x} \tag{8}$$

where $P_m(s) = \frac{P(s)}{s-\alpha_m}$.

Comparing with this we have in (7):

$$P(s) = (s-1)(s^k - y^k)$$

$$= (s-1)(s-\omega_0 y)(s-\omega_1 y)\cdots(s-\omega_{k-1} y)$$

so that $\alpha_1 = 1$, $\alpha_{m+2} = \omega_m y$, $m=0,1,\ldots,k-1$

$\alpha_i \neq \alpha_j$ for $i \neq j$ since $y \neq 1$ by assumption. Where $\omega_0, \omega_1, \ldots, \omega_{k-1}$ are the roots of $z^k - 1 = 0$

$$Q(s) = y^{rk}\left[s^k + s^{\nu-rk}(y^k-1) - y^k\right]$$

$$P_m(s) = \frac{(s-1)(s^k - y^k)}{s - \alpha_m}$$

$$P_1(\alpha_1) = 1 - y^k$$

$$P_{m+2}(\alpha_{m+2}) = (\omega_m y-1)\, y^{k-1}(\omega_m-\omega_0)(\omega_m-\omega_1)\ \ldots$$

$$(\omega_m-\omega_{m-1})(\omega_m-\omega_{m+1})\ \ldots\ (\omega_m-\omega_{k-1})$$

$$= \frac{k}{\omega_m}(\omega_m y-1)y^{k-1},\quad m=0,1,\ldots,k-1$$

where the simplification is obtained from the properties of the roots of the equation $z^k - 1 = 0$.

$$Q(\alpha_1) = 0$$

$$Q(\alpha_{m+2}) = (\omega_m y)^\nu (y^k-1), \; m=0,1,\ldots,k-1.$$

Hence using (8) the inverse transform $f(x,y)$ of $\hat{f}(s,y)$ is given by:

$$f(x,y) = \sum_{m=1}^{k+1} \frac{Q(\alpha_m)}{P_m(\alpha_m)} e^{\alpha_m x}$$

$$= \frac{1}{k} \sum_{m=0}^{k-1} \frac{\omega_m(\omega_m y)^\nu (y^k-1)}{(\omega_m y-1)\, y^{k-1}} e^{\omega_m xy}$$

This is independent of r and hence the result is true for all $\nu \geq 0$.

APPENDIX B

PROPERTIES OF THE TABOO PROBABILITIES ${}_oQ_{ij}^{(n)}(\cdot)$ FOR THE TANDEM QUEUE WITH ZERO SWITCHING

(The results and proofs of this appendix closely follow Neuts (1969))

Starting with the semi-Markov sequence $\{\xi_n, T_n, n \geq 0\}$ defined in (1.3) we define the taboo probabilities ${}_oQ_{ij}^{(n)}(\cdot)$ as:

$${}_oQ_{ij}^{(0)}(x) = \delta_{ij}\, U(x)$$

(1) $${}_oQ_{ij}^{(1)}(x) \equiv Q_{ij}(x) = P\{\xi_n = j,\ T_n \leq x \mid \xi_{n-1} = i\}$$

and

$${}_oQ_{ij}^{(n)}(x) = P\{T_1 + \ldots + T_n \leq x,\ \xi_n = j,\ \xi_\nu \neq 0, \nu = 1,\ldots,n-1 \mid \xi_o = i\}$$

$$n \geq 1\ ,$$

Let ${}_oq_{ij}^{(n)}(s)$ be the L.S.T of ${}_oQ_{ij}^{(n)}(x)$ and

(2) $${}_oq_i^{(n)}(s,z) = \sum_{j=0}^{\infty} {}_oq_{ij}^{(n)}(s)\, z^j\ ,\ |z| \leq 1,$$

Further we denote:

(3) $${}_om_{ij}(s) = \sum_{n=1}^{\infty} {}_oq_{ij}^{(n)}(s) \quad , \ i \geq 0,$$

(4) $$_om_{oj}(s) = \frac{\lambda}{\lambda+s}\ {_om_{1j}}(s)$$

It is seen that:

(5) $$\sum_{j=1}^{\infty} {_om_{ij}}(s)\, z^j = \sum_{n=1}^{\infty} \left[{_oq_i^{(n)}}(s,z) - {_oq_i^{(n)}}(s,0)\right]$$

We define the following sequence of functions:

(6)
$$a_o(s,z) = z$$
$$a_n(s,z) = \gamma_1\{s,h_2[s+\lambda-\lambda a_{n-1}(s,z)]\}\ ,\ n \geq 1\ ,$$

where $\gamma_1(\cdot,\cdot)$ is defined in (1.12).

Throughout this appendix we use the following notations:

$$a_n'(0,1) = \frac{\partial}{\partial z} a_n(0,z)\Big]_{z=1}$$

$$a_n''(0,1) = \frac{\partial^2}{\partial z^2} a_n(0,z)\Big]_{z=1}$$

$$\gamma_1'(0,1) = \frac{\partial}{\partial z} \gamma_1(0,z)\Big]_{z=1}$$

$$\gamma_1''(0,1) = \frac{\partial^2}{\partial z^2} \gamma_1(0,z)\Big]_{z=1}$$

Lemma 1

If $1-\lambda\,\alpha_1 - \lambda\,\alpha_2 > 0$ then:

(7) $$\sum_{n=1}^{\infty} a_n'(0,1) = \frac{\lambda\,\alpha_2}{1-\lambda\,\alpha_1-\lambda\,\alpha_2}$$

and

$$(8) \qquad \sum_{n=1}^{\infty} a_n''(0,1) = \lambda^2\Big\{(1-\lambda\alpha_1-\lambda\alpha_2)\Big[1 - \Big(\frac{\lambda\,\alpha_2}{1-\lambda\alpha_1}\Big)^2\Big]\Big\}^{-1}$$

$$\Big\{\Big(\frac{\lambda\alpha_2}{1-\lambda\alpha_1}\Big)^2 \beta_1 + \beta_2 + 2\Big(\frac{\lambda\alpha_2}{1-\lambda\alpha_1}\Big)\alpha_1\,\alpha_2\Big\}$$

<u>Proof</u>:

From (6) we obtain:

$$a_n'(0,1) = \lambda\,\alpha_2\,\gamma_1'(0,1)\,a_{n-1}'(0,1)\ ,\ n \geq 1\ ,$$

Successive substitution yields:

$$(9) \qquad a_n'(0,1) = \Big[\lambda\,\alpha_2\,\gamma_1'(0,1)\Big]^n a_o'(0,1)$$

$$= \Big(\frac{\lambda\,\alpha_2}{1-\lambda\,\alpha_1}\Big)^n$$

since $\gamma_1'(0,1) = \dfrac{1}{1-\lambda\,\alpha_1}$ by (1.20)

Hence (7) follows for $\dfrac{\lambda\alpha_2}{1-\lambda\alpha_1} < 1$.

Similarly differentiating (6) twice and simplifying results in equation (8).

<u>Lemma 2</u>

For $R(s) > 0$ and $i \geq 1$,

$$(10) \qquad {}_oq_i^{(0)}(s,z) = a_o^i(s,z)$$

$${}_oq_i^{(n)}(s,z) = a_n^i(s,z) - a_{n-1}^i(s,0),\ n \geq 1,$$

<u>Proof</u>:

From (2.3) we have:

$$(11) \qquad q_i(s,z) = \gamma_1^i\{s,h_2(s+\lambda-\lambda z)\}$$
$$= a_1^i(s,z)$$

Again,

$$q_i[s,a_1(s,z)] = \gamma_1^i\{s,h_2[s+\lambda-\lambda a_1(s,z)]\}$$
$$= a_2^i(s,z)$$

and

$$(12) \qquad {}_oq_i^{(n)}\Big[s,a_1(s,z)\Big] = \sum_{\nu=1}^{\infty} {}_oq_{i\nu}^{(n-1)}(s)\; q_\nu[s,a_1(s,z)]$$

$$= \sum_{\nu=1}^{\infty} {}_oq_{i\nu}^{(n-1)}(s)\; a_2^\nu(s,z)$$

$$= {}_oq_i^{(n-1)}[s,a_2(s,z)] - {}_oq_i^{(n-1)}(s,0)$$

$$(13) \qquad {}_oq_i^{(n+1)}(s,z) = \sum_{\nu=1}^{\infty} {}_oq_{i\nu}^{(n)}(s)\; q_\nu(s,z)$$

$$= \sum_{\nu=1}^{\infty} {}_oq_{i\nu}^{(n)}(s)\; a_1^\nu(s,z)$$

$$= {}_oq_i^{(n)}[s,a_1(s,z)] - {}_oq_i^{(n)}(s,0)$$

Setting $z = 0$ in (13) leads to:

$$(14) \qquad {}_oq_i^{(n)}(s,0) = {}_oq_i^{(n-1)}[s,a_1(s,0)] - {}_oq_i^{(n-1)}(s,0)$$

Substitution of (12) and (14) in (13) yields:

$${}_{o}q_i^{(n+1)}(s,z) = {}_{o}q_i^{(n-1)}[s,a_2(s,z)] - {}_{o}q_i^{(n-1)}[s,a_1(s,0)]$$

Successive similar substitution gives:

$$(15)\quad {}_{o}q_i^{(n+1)}(s,z) = {}_{o}q_i^{(1)}[s,a_n(s,z)] - {}_{o}q_i^{(1)}[s,a_{n-1}(s,0)]$$

$$= \gamma_1^i\{s,h_2[s+\lambda-\lambda a_n(s,z)]\}$$

$$- \gamma_1^i\{s,h_2[s+\lambda-\lambda a_{n-1}(s,0)]\}$$

$$= a_{n+1}^i(s,z) - a_n^i(s,0)$$

From (10) and (15) we have:

$${}_{o}q_i^{(n)}(s,z) = a_n^i(s,z) - a_{n-1}^i(s,0),\ n \geq 1,$$

For n = 0,

$${}_{o}q_i^{(0)}(s,z) = z^i$$

$$= a_o^i(s,z)$$

<u>Lemma 3</u>

For $s \geq 0$ and $i \geq 1$,

$$(16)\qquad \lim_{n\to\infty} a_n(s,0) = {}_{o}m_{1o}(s) = \gamma(s)$$

<u>Proof:</u>

From lemma 2 we have:

$${}_{o}q_1^{(n)}(s,z) = a_n(s,z) - a_{n-1}(s,0),\ n \geq 1,$$

Hence:

(17) $$\sum_{n=1}^{N} {}_0q_1^{(n)}(s,0) = a_N(s,0)$$

The left side of (17) is the L.S.T. of the probability $A_N(x)$, where $A_N(x)$ is the probability that a busy period with one customer initially lasts for at most N cycles of tasks and has a duration at most x. It can be argued as in Neuts (1969):

$$A_N(x) \le A_{N+1}(x) < 1,$$

which implies that the transforms $a_N(s,0)$ is increasing in N for $s \ge 0$. Hence by Helly-Bray theorem (Theorem 2, Appendix D), $a_N(s,0)$ converges to the L.S.T. of a probability mass function. That is:

$$\lim_{N \to \infty} a_N(s,0) = \sum_{n=1}^{\infty} {}_0q_1^{(n)}(s,0)$$

$$= {}_0m_{1o}(s) = \gamma(s) \text{ by } (1.35)$$

<u>Lemma 4</u>

If $\sum_{n=1}^{\infty} {}_0q_1^{(n)}(s,0)\, \omega^n = \gamma(s,\omega)$, $|\omega| \le 1$,

then:

(18) $$\sum_{n=1}^{\infty} {}_0q_i^{(n)}(s,0)\omega^n = \gamma^i(s,\omega),\ i \ge 1,$$

Proof:

Analogous to the proof of lemma 1.2.

Lemma 5

If $R(s) \geq 0$ and $|\omega| \leq 1$, then $z = \gamma(s,\omega)$ is a root of the equation:

$$(19) \qquad z = \omega\, \gamma_1\{s, h_2(s+\lambda-\lambda z)\}\ , \quad |z| \leq 1,$$

Further $z = \gamma(s,\omega)$ is the only root of this equation in the unit circle $|z| < 1$ if $R(s) \geq 0$ and $|\omega| < 1$ or $R(s) > 0$ and $|\omega| \leq 1$ or $R(s) \geq 0$, $|\omega| \leq 1$ and $1 - \lambda\,\alpha_1 - \lambda\,\alpha_2 < 0$.

Proof:

Consider the recurrence relation:

$$(20) \qquad {}_o q_{1o}^{(n+1)}(s) = \sum_{\nu=1}^{\infty} q_{1\nu}(s)\ {}_o q_{\nu o}^{(n)}(s)\ , \quad n \geq 1,$$

which gives:

$$\sum_{n=1}^{\infty} \omega^{n+1}\ {}_o q_{1o}^{(n+1)}(s) = \omega \sum_{\nu=1}^{\infty} q_{1\nu}(s) \sum_{n=1}^{\infty} \omega^{n}\ {}_o q_{\nu o}^{(n)}(s)$$

$$= \omega \sum_{\nu=1}^{\infty} q_{1\nu}(s)\ \gamma^{\nu}(s,\omega)$$

(by lemma 4)

$$(21) \qquad = \omega\, \{q_1[s,\gamma(s,\omega)] - q_{1o}(s)\}$$

That is:

$$(22) \qquad \gamma(s,\omega) = \sum_{n=1}^{\infty} \omega^{n}\ {}_o q_{1o}^{(n)}(s) = \omega\, q_1[s,\gamma(s,\omega)]$$

$$= \omega\, \gamma_1\{s, h_2[s+\lambda-\lambda\gamma(s,\omega)]\}$$

(by equation (2.3))

which proves the first part of the lemma.

The second part follows from Rouché's theorem (Theorem 3, Appendix D). For a complete proof we refer to Takács (1962), p. 48.

Lemma 6

For $R(s) \geq 0$, $z = \gamma(s)$ is a root of the equation:

(23) $$z = \gamma_1\{s, h_2(s+\lambda-\lambda z)\}\ , \qquad |z| \leq 1\ ,$$

Further $\gamma(0)$ is the smallest positive real root of the equation:

(24) $$\theta = \gamma_1\{0,\ h_2(\lambda-\lambda\theta)\}$$

and if $1-\lambda\,\alpha_1 - \lambda\,\alpha_2 < 0$ then $\theta < 1$, if $1-\lambda\,\alpha_1 - \lambda\,\alpha_2 \geq 0$ then $\theta = 1$.

Proof:

Proof of the first part is similar to that of lemma 5, by taking $\omega = 1$.

For the second part, the proof is analogous to that of lemma 2 in Neuts (1969). For completeness we repeat that proof here, since our functional equation is different from that in Neuts (1969).

Consider the graphs of:

$$y = x \text{ and } y = \gamma_1\{0, h_2(\lambda-\lambda x)\}$$

and consider the increasing sequence of points whose abscissae are $a_n(0,0)$. At the point $x = a_n(0,0)$ we have:

$$y = \gamma_1\{0,h_2[\lambda-\lambda a_n(0,0)]\} = a_{n+1}(0,0)$$

and $\lim_{n\to\infty} a_n(0,0) = \gamma(0)$ which implies that $\gamma(0)$ is the smallest positive real root of (24).

If $1-\lambda\alpha_1 - \lambda\alpha_2 < 0$ then from lemma 5 it follows that $\theta < 1$. If $1-\lambda\alpha_1 - \lambda\alpha_2 > 0$, then the graph of $y = \gamma_1\{0,h_2(\lambda-\lambda x)\}$ does not intersect the line $y = x$ in $[0,1)$ so that $\theta = 1$ is the only root of (24).

Remark:

From lemmas 5 and 6 it follows that if $1-\lambda\alpha_1 - \lambda\alpha_2 > 0$ then:

(25) $$-\gamma'(0) = \frac{\alpha_1 + \alpha_2}{1 - \lambda\alpha_1 - \lambda\alpha_2}$$

(26) $$\gamma''(0) = \frac{\beta_1 + 2\alpha_1\alpha_2 + \beta_2}{(1-\lambda\alpha_1 - \lambda\alpha_2)^3}$$

(27) $$\frac{\partial}{\partial\omega}\gamma(0,\omega)\Big]_{\omega=0} \equiv \gamma'(0,1) = \frac{1 - \lambda\alpha_1}{1 - \lambda\alpha_1 - \lambda\alpha_2}$$

Lemma 7

If $s \geq 0$, $0 \leq z \leq 1$ and $1-\lambda\alpha_1-\lambda\alpha_2 > 0$ then:

(28) $$\sum_{n=1}^{\infty}\left[a_n(s,z) - a_n(s,0)\right] < \sum_{n=1}^{\infty}\left[1 - a_n(0,0)\right]$$

$$< \frac{\lambda\alpha_2}{1 - \lambda\alpha_1 - \lambda\alpha_2}$$

Proof:

The summands $a_n(s,z) - a_n(s,0)$ is a monotonic increasing function of z and a decreasing function of s. Hence by setting $s = 0$ and $z = 1$ in the summands we get the first part of the inequality (28).

It can be shown that $a_n''(0,z) > 0$ for all z in [0,1], so that $a_n(0,z)$ is a convex function in [0,1] and its graph lies entirely above the tangent at $z = 1$. This tangent has an intercept:

$$a_n(0,1) - a_n'(0,1) = 1 - \left(\frac{\lambda\,\alpha_2}{1-\lambda\alpha_1}\right)^n$$

where the value of $a_n'(0,1)$ is taken from (9).

Hence $a_n(0,0) > 1 - \left(\frac{\lambda\,\alpha_2}{1-\lambda\alpha_1}\right)^n$ which proves the lemma completely.

Limiting Properties of the Semi-Markov Sequence. The limiting properties of the semi-Markov sequence defined in (1.3) is studied through the following theorems.

Theorem 1

If $\lim_{n\to\infty} P\{\xi_n=j \mid \xi_o=i\} = \beta_j$, $j \geq 0$, then $\beta_j = 0$ for all j if $1-\lambda\alpha_1 - \lambda\alpha_2 < 0$. If $1-\lambda\alpha_1-\lambda\alpha_2 \geq 0$ and $0 \leq z \leq 1$, then

$$(29) \qquad B(z) \equiv \sum_{j=0}^{\infty} \beta_j\, z^j = 1 - \beta_o \sum_{n=1}^{\infty} [1 - a_n(0,z)]$$

where β_o is given by:

$$(30) \qquad \beta_o^{-1} = \sum_{n=0}^{\infty} [1 - a_n(0,0)]$$

and $\beta_j > 0$ for all j.

Proof:

The stationarity equations for the imbedded Markov sequence $\{\xi_n\}$ are:

(31)

$$\beta_j = \sum_{r=1}^{\infty} \beta_r \sum_{\nu=1}^{\infty} \int_0^{\infty} \int_u^{\infty} d\, G_{ro}^{(\nu)}(u)\, e^{-\lambda(v-u)} \frac{[\lambda(v-u)]^j}{j!}\, dH_2^{(\nu)}(v-u)$$

$$+ \beta_o \sum_{\nu=1}^{\infty} \int_0^{\infty} \int_u^{\infty} d\, G_{1o}^{(\nu)}(u)\, e^{-\lambda(v-u)} \frac{[\lambda(v-u)]^j}{j!}\, d\, H_2^{(\nu)}(v-u), \; j \geq 0,$$

The first term is obtained by considering $r \geq 1$ customers in unit 1 followed by a 1-task and a 2-task. The second term is obtained by considering an idle period followed by a 1-task and a 2-task. Equation (31) shows that all β_j are strictly positive if and only if β_o is strictly positive.

From (31) we also obtain:

$$B(z) = \sum_{r=1}^{\infty} \beta_r \sum_{\nu=1}^{\infty} g_{ro}^{(\nu)}(0) h_2^{\nu}(\lambda-\lambda z) + \beta_o \sum_{\nu=1}^{\infty} g_{1o}^{(\nu)}(0) h_2^{\nu}(\lambda-\lambda z)$$

$$= \sum_{r=1}^{\infty} \beta_r\, \gamma_1^r\{0,h_2(\lambda-\lambda z)\} + \beta_o\, \gamma_1\{0,h_2(\lambda-\lambda z)\}$$

$$= B(\gamma_1\{0,h_2(\lambda-\lambda z)\}) - \beta_o[1 - \gamma_1\{0,h_2(\lambda-\lambda z)\}]$$

That is:

(32) $$B(\gamma_1\{0,h_2(\lambda-\lambda z)\}) - B(z) = \beta_o[1 - \gamma_1\{0,h_2(\lambda-\lambda z)\}]$$

If $1-\lambda\alpha_1-\lambda\alpha_2 < 0$ then from lemma 6, $z = \gamma(0) < 1$ is a root of the equation $z = \gamma_1\{0,h_2(\lambda-\lambda z)\}$. Setting $z = \gamma(0)$ in (32) we get $\beta_o = 0$ which implies and implied by $\beta_j = 0$ for all j. If $1-\lambda\alpha_1-\lambda\alpha_2 \geq 0$, we replace z in (32) by $a_r(0,z)$ $r = 0,1,\ldots,n-1$ and add the resulting equations to get:

$$B(a_n(0,z)) - B(a_o(0,z)) = \beta_o \sum_{r=1}^{n} [1 - a_r(0,z)] \tag{33}$$

Letting $n \to \infty$ and noting that $a_n(0,z) \to 1$ as $n \to \infty$ for every z in $[0,1]$ we have:

$$B(1) - B(z) = \beta_o \sum_{n=1}^{\infty} [1 - a_n(0,z)] \tag{34}$$

which proves (29) since $B(1) = 1$. Finally equation (30) is obtained by setting $z = 0$ in (34).

Theorem 2

If $0 \leq z \leq 1$ and $1-\lambda\alpha_1-\lambda\alpha_2 \geq 0$ then,

$$\sum_{j=1}^{\infty} {}_o m_{1j}(0)\, z^j = \beta_o^{-1} \sum_{j=1}^{\infty} \beta_j\, z^j \tag{35}$$

Proof:

From (5) we have:

$$\sum_{j=1}^{\infty} {}_o m_{1j}(0)\, z^j = \sum_{n=1}^{\infty} \left[{}_o q_1^{(n)}(0,z) - {}_o q_1^{(n)}(0,0) \right]$$

$$= \sum_{n=1}^{\infty} \left[a_n(0,z) - a_n(0,0) \right]$$

$$= \sum_{n=1}^{\infty} [1-a_n(0,0)] - \sum_{n=1}^{\infty} [1-a_n(0,z)]$$

(This rearrangement is allowed by lemma 7)

$$= (\beta_o^{-1} - 1) - \beta_o^{-1}[1 - B(z)]$$

$$= \beta_o^{-1} B(z) - 1 = \beta_o^{-1} \sum_{j=1}^{\infty} \beta_j z^j$$

<u>Theorem 3</u>

If $s \geq 0$, $0 \leq z \leq 1$ and $1-\lambda\alpha_1-\lambda\alpha_2 \geq 0$, then:

$$(36) \qquad \sum_{n=1}^{\infty} {}_o q_1^{(n)}(s,z) < \frac{1}{\beta_o} < \frac{1-\lambda\alpha_1}{1-\lambda\alpha_1-\lambda\alpha_2}$$

<u>Proof:</u>

For $s \geq 0$ and $0 \leq z \leq 1$:

$$\sum_{n=1}^{\infty} {}_o q_1^{(n)}(s,z) < \sum_{n=1}^{\infty} {}_o q_1^{(n)}(0,1)$$

$$= \sum_{j=0}^{\infty} {}_o m_{1j}(0)$$

$$= 1 + \sum_{j=1}^{\infty} {}_o m_{1j}(0)$$

$$= 1 + \beta_o^{-1} \sum_{j=1}^{\infty} \beta_j \quad \text{(by Theorem 2)}$$

$$= \beta_o^{-1} \text{ , since } \sum_{j=0}^{\infty} \beta_j = 1,$$

$$= 1 + \sum_{n=1}^{\infty} [1 - a_n(0,0)] \text{ , by Theorem 1}$$

$$< 1 + \frac{\lambda \alpha_2}{1 - \lambda \alpha_1 - \lambda \alpha_2} = \frac{1 - \lambda \alpha_1}{1 - \lambda \alpha_1 - \lambda \alpha_2}$$

(by lemma 7)

<u>Theorem 4</u>

If $1 - \lambda \alpha_1 - \lambda \alpha_2 \geq 0$ then:

(37) $$\sum_{j=1}^{\infty} j \; {}_o m_{1j}(0) = \frac{\lambda \alpha_2}{1 - \lambda \alpha_1 - \lambda \alpha_2}$$

(38) $$\sum_{j=1}^{\infty} j(j-1)\,{}_o m_{1j}(0) = \lambda^2 \left\{ (1-\lambda\alpha_1-\lambda\alpha_2) \left[1 - \left(\frac{\lambda\alpha_2}{1-\lambda\alpha_1}\right)^2 \right] \right\}^{-1} \left\{ \left(\frac{\lambda\alpha_2}{1-\lambda\alpha_1}\right)^2 \beta_1 + \beta_2 + 2\left(\frac{\lambda\alpha_2}{1-\lambda\alpha_1}\right)\alpha_1\alpha_2 \right\}$$

<u>Proof</u>:

Equations (5) and (10) give:

(39) $$\sum_{j=1}^{\infty} {}_o m_{1j}(0)\, z^j = \sum_{n=1}^{\infty} \left[a_n(0,z) - a_n(0,0) \right]$$

Differentiation with respect to z gives:

(40) $$\sum_{j=1}^{\infty} j \; {}_o m_{1j}(0) = \sum_{n=1}^{\infty} a_n'(0,1)$$

where term by term differentiation is valid by lemma 7.

Similarly,

$$(41) \qquad \sum_{j=1}^{\infty} j(j-1)\ {}_{o}m_{1j}(0) = \sum_{n=1}^{\infty} a_n''(0,1)$$

Substitution of lemma 1 in (40) and (41) proves the theorem.

APPENDIX C

To facilitate reading Chapter IV we state and prove the following lemmas which are essentially in Neuts (1969).

Starting with the semi-Markov sequence $\{\xi(T_n), T_{n+1} - T_n, n \geq 0\}$ defined in (4.1) we define the taboo probabilities ${}_oQ_{ij}^{(n)}(\cdot)$ as:

$$ {}_oQ_{ij}^{(0)}(x) = \delta_{ij}\, U(x) $$

(1) $$ {}_oQ_{ij}^{(1)}(x) \equiv Q_{ij}(x) = P\{\xi(T_n)=j,\ T_{n+1}-T_n \leq x \mid \xi(T_{n-1})=i\} $$

and

$$ {}_oQ_{ij}^{(n)}(x) = P\{T_n \leq x,\ \xi(T_n)=j, \xi(T_\nu)\neq 0, \nu=1,\ldots,n-1 \mid \xi(T_o)=i\} $$

Let ${}_oq_{ij}^{(n)}(s)$ be the L.S.T. of ${}_oQ_{ij}^{(n)}(x)$ and

(2) $$ {}_oq_i^{(n)}(s,z) = \sum_{j=0}^{\infty} {}_oq_{ij}^{(n)}(s)\, z^j, \quad |z| \leq 1, $$

We define the following sequence of functions:

(3) $$ \begin{aligned} h_o(s,z) &= z \\ h_n(s,z) &= h[s+\lambda - \lambda\, h_{n-1}(s,z)]\ ,\ n \geq 1 \end{aligned} $$

Further we denote:

$$h_n'(0,1) = \frac{\partial}{\partial z} h_n(0,z)\Big]_{z=1}$$

$$h_n''(0,1) = \frac{\partial^2}{\partial z^2} h_n(0,z)\Big]_{z=1}$$

$$h_n'''(0,1) = \frac{\partial^3}{\partial z^3} h_n(0,z)\Big]_{z=1}$$

<u>Lemma 1</u>

If $1 - \lambda \alpha > 0$, then:

(4) $$\sum_{n=0}^{\infty} h_n'(0,1) = \frac{1}{1-\lambda\alpha}$$

(5) $$\sum_{n=0}^{\infty} h_n''(0,1) = \frac{\lambda^2 \beta}{(1-\lambda\alpha)\,(1-\lambda^2\alpha^2)}$$

(6) $$\sum_{n=0}^{\infty} h_n'''(0,1) = \frac{1}{(1-\lambda\alpha)}\left[\frac{\lambda^3\gamma}{1-\lambda^3\alpha^3} + \frac{3\,\lambda^5\,\alpha\,\beta^2}{(1-\lambda^2\alpha^2)(1-\lambda^3\alpha^3)}\right]$$

where α, β and γ are defined in Chapter IV.

The proof is similar to that of lemma 1 in Appendix B.

<u>Lemma 2</u>

For $R(s) > 0$ and $i \geq 1$,

(7) $${}_0q_i^{(0)}(s,z) = h_o^i(s,z)$$

$${}_0q_i^{(n)}(s,z) = h_n^i(s,z) - h_{n-1}^i(s,0),\ n \geq 1$$

Again the proof of this lemma is analogous to the proof of lemma 2 in Appendix B.

<u>Lemma 3</u>

If $s \geq 0$, $0 \leq z \leq 1$ and $1 - \lambda\alpha > 0$, then:

$$(8) \qquad \sum_{n=1}^{\infty} [h_n(s,z)-h_n(0,0)] < \sum_{n=1}^{\infty} [1-h_n(0,0)] < \frac{\lambda\alpha}{1-\lambda\alpha}$$

For the proof we refer to the parallel proof of lemma 7 in Appendix B.

For further properties of the taboo probabilities defined in this appendix we refer to Neuts (1969) where they are extensively treated.

APPENDIX D

SOME WELL KNOWN THEOREMS USED IN THE TEXT

Theorem 1: ZYGMUND'S THEOREM

Let $\{F_n(x)\}$ be a sequence of distribution functions all vanishing for $x \leq 0$ and let

(1) $$\phi_n(\omega) = \int_0^\infty e^{i\omega x}\, dF_n(x), \quad -\infty < \omega < \infty .$$

If the functions $\phi_n(\omega)$ tend to a limit in an interval around $\omega = 0$, and if the limiting function is continuous at $\omega = 0$, then there is a distribution function $F(x)$ such that $F_n(x)$ tends to $F(x)$ at every point of continuity of $F(x)$.

[Ref: Zygmund (1951)]

Theorem 2: HELLY-BRAY THEOREM

If $g(x)$ is bounded and continuous when $-\infty < x < \infty$ and the sequence of distribution functions $F_n(x)$ converges to a distribution function $F(x)$, then:

(2) $$\lim_{n \to \infty} \int_{-\infty}^{\infty} g(x)\, d\, F_n(x) = \int_{-\infty}^{\infty} g(x)\, dF(x)$$

[Ref: Loéve (1963)]

Theorem 3: ROUCHE'S THEOREM

If $f(z)$ and $g(z)$ are regular inside and on a closed contour C, and $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.

[Ref: Whittaker and Watson (1952)]

Theorem 4: SMITH'S KEY RENEWAL THEOREM

If $M(t)$ is the expected number of renewals in $(0,t]$, $Q(\cdot)$ is a positive integrable and decreasing function, then:

$$\int_0^t Q(t-u)\, dM(u) \rightarrow \frac{1}{\mu} \int_0^\infty Q(u)\, du \tag{3}$$

where μ is the mean renewal time

[Ref: Smith (1958)]

Theorem 5: A TAUBERIAN THEOREM

If $M(t)$ is non-decreasing and such that $m(s) = \int_0^\infty e^{-st}\, dM(t)$ converges for $R(s) > 0$, and if for some non-negative number α, $\lim_{s \to 0} s^\alpha m(s) = c$, then:

$$\lim_{t \to \infty} \frac{M(t)}{t^\alpha} = \frac{c}{\Gamma(\alpha+1)}$$

[Ref: Widder (1947)]

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13. ABSTRACT This thesis deals with three priority queues. Chapters I and II treat a queueing model with two service units in tandem and a single server alternating between them. Chapter III deals with two independent service units with a single server serving alternately between them and Chapter IV treats a single server M|G|1 queue with a priority rule based on the ranking of the service times. In Chapter I the server serves the two service units alternately with a non-zero switching rule in unit 1 and a zero switching rule in unit 2. The case of zero switching rule for unit 1 is dealt in Chapter II. In both cases the distributions of busy period, virtual waiting time and queue length and their ergodic properties are studied in terms of Laplace transforms. In Chapter III we consider the alternating priority queues with a non-zero switching in each unit. Distributions of busy period and queue length are discussed. In Chapter IV we study the virtual waiting time process of an M|G|1 queue under this priority rule: within each generation customers are served in the order of shortest (or longest) service times. Here we also study the limiting behavior of the virtual waiting time, and compare the means of the limiting distributions with those of first come, first served discipline. Applications of the different priority models are discussed at the end of Chapters II, III and IV.

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